How do People discount the Very Distant Future?
A Theoretical Approach

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Abstract

In this paper, we consider a Decision Maker facing alternatives defined on very distant future, that is, outcomes that may occur after a very long delay. Hyperbolic Discounted Utility has been extensively used to describe these decisions since it confers greater importance to very distant future and is less responsive to change in discount rate than Exponential Discounted Utility (Harvey 1994). Following a recent work by Bleichrodt et al. (2009), we define an axiomatic property, called "asymptotic patience", suggesting that hyperbolic discounting may be not relevant to modelize choices defined on very distant future. Asymptotic patience means that, for very distant delays, the decision maker always prefers the larger outcome to the smaller outcome, even if the former is shifted in any more remote future. A decision maker who exhibits asymptotic patience confers greater importance to very distant future. We provide an axiomatic model called Subjective Discounted Utility that generalizes any Discounted Utility model. In particular, the Subjective Discounted Utility allows representation for time preferences consistent with asymptotic patience, for which discount functions give a minimal positive weight to any outcome delayed in very distant future. The Subjective Discounted Utility model also provides a sharp expression to very distant future based on psychological distortion of linear time called time perception.

KEYWORDS:	Discounting, Time Tradeoff, Very Distant Future, Asymptotic Patience, Subjective Discounted Utility
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1. Introduction

Discounted Utility (DU) is the standard model to describe and to explain intertemporal choices made by individual decision makers. The core advantage of DU models is to focus the study of time preferences on properties related to specific discount functions. Exponential Discounted Utility (EDU) has been first used to modelize individual choices (Koopmans 1972).However, Thaler (1981) stretched that cognitive psychology may provide strong behavioral foundations to an alternative DU model called Hyperbolic Discounted Utility (HDU). Indeed, one of the most robust results provided by experimental economics is that, for delays defined as months or as a few years, observed time preferences exhibit decreasing impatience, that is, long term discount rates are lower than short term discount rates (Mazur 1987, Loewenstein and Prelec 1992, Harvey 1995).

DU models are based on an impatience axiom which is a core element of time rationality for standard time horizon. Impatience stipulates that a decision maker (DM) always prefers receiving a gain sooner than later. Impatience rises because of the uncertainty on the receipt of future outcomes and of the DM's inclination to exhibit excitement produced by the prospect of immediate satisfaction (Frederick et al. 2002). A key implication of impatience is the potential occurrence of "time tradeoff", that is, the existence of time preference reversals: initially, a DM always prefers a larger outcome now to a smaller outcome now, but, if the larger outcome is delayed in a sufficient remote future, then the time preference *could* reverse for the smaller outcome. Noor (2011) identified an axiomatic condition, called "strong impatience", implying time tradeoff for any time horizon. In this paper, we wonder about the reasonableness to consider impatience as a core element of time rationality when alternatives are defined on very distant future. The main idea is that, for very distant alternatives – say outcomes delayed on many decades –, uncertainty on the receipt of both outcomes is radical and the excitement associated to immediate satisfaction vanishes at all. As a consequence, far-off delaying expunges any trace of impatience and very distant future may be defined as a time horizon on which there is no time tradeoff anymore. Unfortunately, standard discounting – EDU and HDU – implies strong impatience. Then, we need to define an alternative discounting rule to rationalize the absence of time tradeoff on very distant future. Strong impatience will be substituted by an axiom called "weak impatience", which requires the existence of a delay from which time tradeoff no longer occurs.

In a previous work, Lapied and Renault (2012) have built an axiomatic model, called Subjective Discounted Utility (SDU), that explains the diversity in time discounting by the diversity in individual subjective time perceptions, that is, distortions applied to linear time. In this model, each discount function is related to a single time perception. In particular, we extend SDU for discount functions that exhibit weak impatience. Moreover, another joint-product associated to SDU lies on the analytical expression dedicated to very distant future. Indeed, for SDU, the time tradeoff horizon is allowed to vary from a few months – for myopic decision makers – to many years – for telescopic decision makers –.depending on psychological considerations related to time perception properties.

The paper is organized as follows. The second section provides notations used throughout the paper and introduces the concept of "time tradeoff". A key observation reveals that DU models do not necessarily imply time tradeoff. The third section deals with the discounting of very distant future, considering two distinct cases: strong impatience, when time tradeoff is extended for any delay, and weak impatience, when time tradeoff is dropped for very distant delays. The fourth section presents the SDU model that enables us to axiomatize discount functions consistent with weak impatience and gives sound behavioral interpretation for strong impatience and for weak impatience. Last, building upon the SDU representation of time preferences, the fifth section confers a sharp analytical expression to the very distant future based on time perception's properties. In particular, we propose an axiomatic condition characterizing the specific case studied by Bleichrodt et al. (henceforth BRW) (2009) and called strongly decreasing impatience.

2. Discounted Utility Models

2.1. Notations and Model

We consider a DM facing alternatives defined as timed outcomes $(x,t) \in \Delta = \mathcal{X} \times \mathcal{P}$, where (x,t) designates the receipt of the outcome x after t periods. An outcome may cover a large range of situations: x may be a real number and, in this case, $\mathcal{X} = \mathbb{R}$ or x may be a vector of n components and, in this case, $\mathcal{X} = \mathbb{R}^n$. If the outcome designates welfare such as the quality of environment, then we will assume that x is defined as its monetary equivalent and, if the outcome designates a lottery such as the distribution of uncertain returns on investment, then we will assume that x is defined as its certain equivalent. For convenience, we will restrict our study to gains and we define $x, y \in \mathcal{X}$ such that $0 < x < y$. As usual, $\mathcal{P} = [0; +\infty[$ is the time horizon, with $t_1, t_2 \in \mathcal{P}$ such that $0 \leq t_1 < t_2$. The time horizon both includes current decision period, normalized to 0, and very distant periods. We assume that there is no uncertainty on the timing of outcomes reception[3].

A binary relation $\succsim$ [4], called "time preference", is defined on $\Delta \times \Delta$ and we assume that $\succsim$ is a continuous weak order on $\Delta \times \Delta$ that verifies separability[5], that is, a common assumption in the literature that enables to clearly dissociate the "outcome effect" from the "time effect" on time preferences. The outcome effect is embodied in a monotonicity axiom, that is, $\forall t \in \mathcal{P}, y > x \Longrightarrow (y,t) \succ (x,t)$. When alternatives are equally delayed, time preferences always turn for the higher outcome and positive monotonicity for outcomes holds. The time effect is embodied in an impatience axiom, that is, $\forall x \in \mathcal{X}, t_2 > t_1 \Longrightarrow (x,t_1) \succ (x,t_2)$. Impatience suggests that the DM always prefers to receive a gain sooner than later and leads to negative monotonicity for time. Continuous weak order, separability, monotonicity and impatience are very standard assumptions and Fishburn and Rubinstein (1982) showed that this set of assumptions leads to a representation of time preferences by discounted utility functions.

Definition 2.1 (Discounted Utility)

The Discounted Utility (DU) is a separable utility function such that, $\forall (x,t) \in \Delta$, $v(x,t) = \phi(t).u(x)$ where ϕ is a discount function and u is a stationary utility function, that is:

- $\phi : \mathcal{P} \to \mathcal{P}'$ [6] *with $\phi(0) = 1$, ϕ is strictly decreasing with t, and, for any t in $\mathcal{P}$, $\phi(t) > 0$.*
- $u : \mathcal{X} \to \mathcal{X}$ *with $u(0) = 0$ and u is strictly increasing with t.*

Assumption A0

Time preferences $\succsim$ are represented by DU functions, that is:

$$\forall (x,t), (x',t') \in \Delta,\ (x,t) \succsim (x',t') \Leftrightarrow \phi(t).u(x) \geq \phi(t').u(x')$$

The core advantage of DU representations is to focus the analysis on discount function properties. However, doing this, DU representations exclude behavioral structures such as magnitude dependent preferences (Noor 2011), when the time effect is outcome dependent, that is, $v(x,t) = m(x,t).u(x)$.The most famous expressions of DU representations are Exponential Discounted Utility (EDU) with $\phi(t) = \lambda^t$ $(0 < \lambda < 1)$ (Koopmans 1972) and Hyperbolic Discounted

[3] For a model with uncertainty on the timing of the receipt of outcomes, see Dasgupta and Maskin (2005).

[4] The relations $\succ$ and $\sim$ are respectively the asymmetric and symmetric parts of the binary relation.

[5] For brevity, we do not restate this set of assumptions and invite the reader to refer to Fishburn and Rubinstein (1982).

[6] Formally, the set $\mathcal{P}'$ is a connected set included into $\mathcal{P}$. Thus, even $\mathcal{P}' = \mathcal{P} = [0; \infty[$ or $\mathcal{P}' = [0; l[\subset \mathcal{P}$ $(l > 0)$.

Utility (HDU) with $\phi(t) = (1 + \alpha t)^{-\beta/\alpha}$ $(0 < \alpha, \beta)$ (Loewenstein and Prelec 1992). EDU, and more recently HDU, have been extensively used to modelize individual behaviors in many applications for intertemporal choices and, in particular, choices defined on very long time horizons.

2.2. Time Discounting

Time discounting expresses the psychological discomfort associated to the waiting of a postponed reward. In spite of the great diversity in individual time discounting (Frederick et al. 2002), most of discount functions used in the literature imply time tradeoff, that is, the possibility that time effect counterbalances outcome effect. Initially, when both alternatives are equally delayed, time preferences are fully explained by the outcome effect and the larger outcome is strictly preferred. But, if the larger outcome is additionally delayed in a sufficient remote future, then time preferences may turn for the sooner and smaller outcome.

Definition 2.2 (Time Tradeoff)

For all outcomes $x, y > 0$, there is time tradeoff for delay t if there exists a finite delay $\tau(t) > 0$ such that

$$(x, t) \sim \big(y, t + \tau(t)\big) \tag{2.1}$$

Time tradeoff stipulates that DM's impatience is so strong that time preferences may reverse for the smaller outcome. The additional time interval $\tau(t)$ is called DM's willingness to wait. The willingness to wait $\tau(t)$ is the maximum amount of time that an individual is willing to wait for an increase in outcome "$y - x$" when both alternatives are initially delayed by t periods. Impatience implies that $\tau(t) > 0$ and we will note $\tau(0) = \tau$ the initial willingness to wait. Note that, under DU, the willingness to wait only depends on time arguments τ and t in the sense that, if $(x, 0) \sim (y, \tau)$ and $(x', 0) \sim (y', \tau)$, then, when alternatives are delayed by t periods, the willingness to wait is equal to $\tau(t)$ in both cases[7]. The willingness to wait is time sensitive if, for $0 \leq t_1 < t_2$, $\tau(t_2)$ is allowed to be different from $\tau(t_1)$. The time premium θ captures such time sensitivity. Indeed, consider time tradeoff for delay t_1 such that $(x, t_1) \sim \big(y, t_2 = t_1 + \tau(t_1)\big)$[8] and assume that time tradeoff holds if both alternatives are delayed by t periods:

$$(x, t_1 + t) \sim (y, t_2 + t + \theta) \tag{2.2}$$

Equation (2.2) is called time tradeoff equation. Combining equation (2.1) defined for delay $t_1 + t$ with the time tradeoff equation, it is easy to show that $\theta = \tau(t_1 + \mathrm{t}) - \tau(t_1)$. Constant impatience holds if $\theta = 0$ $\big(\tau(t_1 + \mathrm{t}) = \tau(t_1)\big)$, that is, $\tau(t)$ is independent from time. Conversely, if $\theta > 0$ (resp $\theta < 0$), then τ is increasing (resp. decreasing) with delay t and impatience is decreasing (resp. increasing) with time, since the DM is willing to wait a greater (resp. smaller) amount of time for an increase in her well-being when both alternatives are delayed in more distant future. Time tradeoff properties related to DU representations are summarized in the following observation.

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[7] This property is directly involved by the separability axiom (see Fishburn and Rubinstein 1982).

[8] Noor (2011) has shown that, for any $t_1, t_2 \geq 0$ and any $y > 0$, there exists a unique $x > 0$ such that $(x, t_1) \sim (y, t_2)$ (Appendix B, Lemma B1).

Observation 2.3

Under DU representation of time preferences:

- (i) *If impatience is constant or increasing with time, then there is time tradeoff for any delay* t.
- (ii) *If there is time tradeoff for delay* t_2*, then there is time tradeoff for any delay* $t_1 \leq t_2$.
- (iii) *Assume impatience is decreasing with time. If there is no time tradeoff for delay* t_1*, then, for any delay* $t_2 \geq t_1$*, there is no time tradeoff anymore.*

First, EDU representation is always consistent with time tradeoff. Moreover, Observation 2.3 stresses that, without any additional assumption than regular DU axioms, there is a prior asymmetry for time tradeoff on the time horizon. Assume time tradeoff for delay t, then the time tradeoff property is fully extended for smaller delays, but not certainly extended for larger delays in the case when impatience is allowed to decrease so strongly that time effect cannot counterbalance outcome effect. Focusing on time preferences exhibiting decreasing impatience, a key joint-product associated to Observation 2.3 is that DU representation does not necessarily imply time tradeoff for all delays. As we will see in the next section, this is precisely the case for CADI discounted utilities ($\phi(t) = e^{r(e^{-ct}-1)}$ defined for $r, c > 0$) introduced by BRW (2009).

3. Discounting the very distant future

In DU models, the impatience axiom is the only source for the raise of time tradeoff. Indeed, for a sufficient large delay t', time preferences may change for the sooner and smaller outcome $(x, t) \succ (y, t + t')$ if and only if the discount function ϕ is "significantly" decreasing with time, that is, the relative depreciation in outcome $(u(y)/u(x))$ is counterbalanced by the relative depreciation in time $(\phi(t)/\phi(t + t'))$. In this case, the DM considers that it does not worth waiting for additional time and time preferences turn for the sooner and smaller outcome. Conversely, if impatience strongly decreases with time in the sense that the relative time depreciation cannot exceed the relative outcome depreciation $(u(y)/u(x) > \phi(t)/\phi(t + t'))$, then there is no time tradeoff anymore. In this section, we will discuss about the reasonableness of the impatience assumption for time preferences defined on very distant future. We argue that delaying alternatives on very long delays leads to expunge any trace of impatience and, consequently, any possibility of time tradeoff.

3.1. Absence of time tradeoff for very distant choices

Empirical findings have provided robust evidence in favor of decreasing impatience with time (Thaler 1981). In particular, the literature on intertemporal choice usually considers that decreasing impatience is relevant to describe choices defined on very long delays. Harvey (1994) advocated for the reasonableness of decreasing impatience since it confers greater importance to very distant outcomes and is less sensible to the choice of the discount rate than constant impatience. To illustrate, consider the following example that describes time preferences exhibiting decreasing impatience:

$$(x, t_1 = \text{one month}) \succsim (y, t_1 = \text{one month } + \Delta = \text{one year})$$

and

$$(y, t_2 = \text{twenty years } + \Delta = \text{one year}) \succ (x, t_2 = \text{twenty years})$$

When alternatives are both delayed by a brief initial delay t_1 (one month), the relative additional time interval Δ (one year) may seem very long (Choice 1) so that the DM is not willing to wait for one additional year to get y instead of x. In this case, the time effect counterbalances the outcome effect because the impatience rate still remains high. On the contrary, when alternatives are both delayed by a very long initial delay t_2 (twenty years), the relative additional time interval Δ (one year) may seem insignificant (Choice 2) compared with t_2 so that the DM is willing to wait for one additional year to get y instead of x. In this case, the DM's impatience has strongly decreased and only matters the outcome effect.

Why could impatience decrease so strongly as alternatives are delayed in distant future? According to Frederick et al. (2002), impatience is motivated by the combination of two distinct factors: the "uncertainty of human life" and "the excitement produced by immediate reward". Basically, the more uncertain the receipt of an outcome is, the less the DM will care about the future. Alternatively, the more the DM exhibits excitement for an immediate reward, the more the discomfort of deferring such a reward is and the less the DM will care about the future. The association of uncertainty on the receipt of outcomes with the excitement for immediate reward produces a psychological cost induced by the delaying of a reward. Thus, for any dated outcome (y, τ), DU assumptions imply that there exists an outcome x such that $x < y$ and $(x, 0) \sim (y, \tau)$. However, assume now that the two previous dated outcomes, $(x, 0)$ and (y, τ), are delayed by a very large delay t. Thus, the receipt of both outcomes is highly uncertain and excitement produced by the prospect of immediate consumption vanishes at all so that time preference $(y, t + \tau + t^{'}) \succ (x, t)$ may hold for any additional delay $t^{'}$. As a consequence, impatience is not yet a core element of time rationality for very distant time horizons.

Definition 3.1 (Very distant future)

> *The very distant future is a time horizon on which there is no time tradeoff, that is, for some $x, y > 0$ and for a given delay t, $\nexists \tau(t) > 0$ such that $(x, t) \sim \big(y, t + \tau(t)\big)$.*

Very distant future is a psychological time horizon. For DM whose impatience rates remain significant, no matter the delay t on which choices are defined, there is no very distant future. Conversely, for DM whose impatience might be totally expunged, the timing of very distant future depends on the rate of decreasing impatience. Let us divide the time horizon $\mathcal{P}$ into two subintervals, then very distant future appears as the complementary part of time tradeoff horizon. For the remaining, we will defined T as the time tradeoff horizon, that is, a delay such that, $\forall t < T$, there is time tradeoff. For exponential discounting, time tradeoff may be extended for any additional delay t (Observation 2.1), so the time tradeoff horizon T is infinite and there is no place for very distant future. Alternatively, if time tradeoff is restricted for moderate delays, then the time tradeoff horizon T is finite and, for all delays $t \geq T$, very distant future holds. In this way, the impatience axiom may be sharply completed for very distant delays in two distinct ways.

Definition 3.2 (Asymptotic Impatience / Asymptotic Patience)

Asymptotic Impatience	$\forall x, y \in \mathcal{X}\ (0 < x < y), \exists t^{'} \in \mathcal{P} : \forall t \in \mathcal{P}, (x, t) \succ (y, t + t^{'})$
Asymptotic Patience	$\forall x, y \in \mathcal{X}\ (0 < x < y), \exists t \in \mathcal{P} :\ \forall t^{'} \in \mathcal{P}, (y, t + t^{'}) \succ (x, t)$

Asymptotic impatience means that, no matter the delay on which are defined the alternatives, impatience still remains substantial to make insignificant any larger outcome, providing that the latter is delayed in a sufficient remote future. We define strong impatience as the combination of impatience and asymptotic impatience (Noor 2011). Alternatively, asymptotic patience means that impatience strongly decreases with time so that the impatience rate falls to zero and time tradeoff cannot occur anymore. We define weak impatience as the combination of impatience and asymptotic patience.

Note that asymptotic patience may be consistent with time tradeoff for small delays. Strong impatience and weak impatience may be related to the nature of impatience in the following way.

Observation 3.3

Constant Impatience or increasing impatience implies strong impatience.
Weak impatience implies that there exists a delay t from which decreasing impatience holds.

Strong impatience is a standard assumption in the literature (Masatlioglu and Ok 2007, Noor 2011) and most of discount functions satisfy strong impatience. The interpretation of Observation 3.3 is more complex for time preferences exhibiting decreasing impatience. If the decrease of the impatience rate is limited, then time tradeoff is extended for any delay and strong impatience holds. As we will see, this is the case for hyperbolic discounting. But, if the decrease in impatience may be arbitrary large, then impatience is totally consumed and there is no time tradeoff anymore. In this sense, asymptotic patience reflects the absence of upper bound on the decrease of impatience.

3.2. Time preferences with asymptotic patience

Among the set of discount functions that satisfy decreasing impatience, is hyperbolic discounting the most relevant discount function to modelize choices defined on very long delays? Abdellaoui et al. (2010) have brought empirical evidence for strongly decreasing impatience. Roughly speaking, time preferences exhibit strongly decreasing impatience if the decrease in impatience could be so strong that the discount rate is allowed to take any value as near as possible to zero. Formally, BRW (2009) identified strongly decreasing impatience as the case when the time premium $\theta(t_1, t_2, t)$[9] is not bounded by the value $t \times (t_2 - t_1)/t_1$.

Definition 3.4 (Strongly Decreasing impatience)

Assume $(x, t_1) \sim (y, t_2)$. Time preferences $\succsim$ exhibit strongly decreasing impatience if there exists an additional delay $t > 0$ such that $(y, t_2 + t + t \times (t_2 - t_1)/t_1) \succsim (x, t_1 + t)$.

BRW (2009) have demonstrated that hyperbolic discounting involves weak decreasing impatience, that is, for any delay $t_1, t_2, t > 0$, we have $\theta(t_1, t_2, t) < t \times (t_2 - t_1)/t_1$ (Observation 3.3, p. 30). Thus, weak decreasing impatience implies strong impatience and hyperbolic discounting is not a good candidate to modelize very distant future since time tradeoff holds for any delay. For instance, it is straightforward that hyperbolic discounting cannot explain the following time preferences:

$$(x, t_1 = \text{one year}) \succ \left(y, t_1 = \text{one year} + \Delta_{t_1} = \text{one month}\right)$$

and

$$\left(y, t_2 = \text{twenty years } + \Delta_{t_2} = \text{three years}\right) \succ (x, t_2 = \text{twenty years})$$

More generally, the next property, called squashing property, allows us to identify the set of discount functions that extend time tradeoff on the entire time horizon.

[9] The time premium $\theta(t_1, t_2, t)$ is defined such that $(x, t_1) \sim (y, t_2)$ and $(x, t_1 + t) \sim \left(y, t_2 + t + \theta(t_1, t_2, t)\right)$.

Definition 3.5 (Squashing Property)

For all delay $t \in \mathcal{P}$ *and* $\forall \varepsilon > 0$, *there exist delays* $t^{'}$ $(t < t^{'})$ *such that:*

$$\frac{\phi(t^{'})}{\phi(t)} < \varepsilon \tag{3.1}$$

The squashing property expresses that, for any distant future, it is possible to find more remote future such that the weight dedicated to later future appears as small as possible compared with the weight associated to sooner future. Intuitively, for any dated outcomes (x, t) and $(y, t\,)$, outcome y, no matter how large is it, can be made unattractive compared with outcome x, no matter how small is it, providing that the larger outcome y is delayed in a sufficient remote future $t^{'}$. In other words, setting $\varepsilon = u(x)/u(y)$, the squashing property states that outcome effect may always be counterbalanced by time effect.

Proposition 3.6

Under DU representation of time preferences, the three following propositions are equivalent:

(i) *Time preferences* $\succsim$ *verify strong impatience.*

(ii) *The discount function* ϕ *is consistent with the squashing property.*

(iii) $lim_{t\to\infty}\, \phi(t) = 0$.

Proposition 3.6 identifies a simple condition on discount functions to characterize strong impatience: time preferences exhibit strong impatience if and only if the discount function confers no weight to arbitrarily late future. Consequently, weak impatience holds if and only if $lim_{t\to\infty}\, \phi(t) = L > 0$ and gathers the set of discount functions which allow minimal weight to any distant future. Indeed, consider that alternatives are defined on very distant future. Then, the relative depreciation $L < 1$ is applied to both alternatives, but, beyond the time tradeoff horizon, the DM behaves as if she does not discount alternatives anymore and time preferences may be modelized from the use of a zero discount rate. In particular, the CADI discount function, presented by BRW (2009), satisfies weak impatience. We introduce an alternative discount function, called DADI (Decreasing Absolute Decreasing Impatience) discounting, that also satisfies weak impatience.

Definition 3.7

For any dated outcome $(x, t) \in \Delta$,

CADI DU is defined by $v(x,t) = \lambda^{1-e^{-ct}} \times u(x)$ $(c > 0)$ *with* $lim_{t\to\infty}\, \phi(t) = \lambda > 0$ [10].

DADI DU is defined by $v(x,t) = \lambda^{[t/(1+at)]} \times u(x)$ $(a > 0)$ *with* $lim_{t\to\infty}\, \phi(t) = \lambda^{1/a} > 0$

In the next section, axiomatic foundations for discount functions consistent with asymptotic patience are provided.

[10] λ is a discount factor such that $0 < \lambda < 1$.

4. The Subjective Discounted Utility

Time discounting reveals the psychological discomfort induced by the waiting of time intervals associated to postponed rewards. Thus, time discounting may be explained from the subjective perception of time intervals, called "time perception". Time perception represents a subjective distortion applied to linear time intervals. The linear time perception expresses no distortion on future time intervals whereas the logarithmic time perception is a subjective transformation of linear time implying that remote time intervals seem shorter than corresponding nearer time intervals. Thus, the time sensitivity of willingness to wait may be fully explained by time perceptions. For instance, consider a logarithmic time perception that reduces a five year linear delay to a one year subjective delay, then, the DM is willing to wait for larger additional time to get the higher outcome. In a previous paper, Lapied and Renault (2012) have built an axiomatic model, called Subjective Discounted Utility (SDU), rationalizing a large range of Discounted Utility models. However, strong impatience was implicitly assumed. As a consequence, the set of Discounted Utilities exhibiting asymptotic patience was excluded from the discussion. In this section, we generalize this previous result by dropping the strong impatience assumption. Doing this, the main theorem of this section rationalizes any Discounted Utility model. Subjective Discounted Utility involves two main advantages. First, the SDU provides axiomatic foundations for any DU representation of time preferences. As we will see, EDU corresponds to SDU with linear time perception, whereas HDU corresponds to SDU with logarithmic time perception. Second, the SDU defines psychological foundations for time discounting based on time perception properties. In particular, asymptotic patience will be captured by a specific property attached to time perception, allowing a clear interpretation for the use of zero discount rates.

4.1. Axiomatic foundation for time discounting

Most of technical tools presented in this section have been previously introduced in Lapied and Renault (2012). We briefly restate the main definitions that allow us to represent time preferences by subjective discounted utility.

Definition 4.1 (Time Perception)

Time perception is a continuous function $\rho : \mathcal{P} \to \mathcal{P}'$ such that:

$(*)\quad \rho(0) = 0$

$(**)\quad 0 \le t_1 < t_2 \implies \rho(t_1) < \rho(t_2)$

Definition 4.2 (Subjective Discounted Utility)

The Subjective Discounted Utility is a discounted utility whose discount function is described as a discount factor λ $(0 < \lambda < 1)$ applied to a time perception ρ: $\forall (x,t) \in \Delta$,

$$v(x,t) = \lambda^{\rho(t)} u(x) \tag{4.1}$$

Time perception conditions only require that there is no distortion for present and that time order is preserved. For all delays $t \in \mathcal{P}$, we define linear time perception as $\rho(t) = t$ and logarithmic time perception as $\rho(t) = \ln(1 + \alpha t)/\alpha$ $(\alpha > 0)$. Note that minimal conditions on time perception insure that ρ is invertible. The SDU representation defines a one-to-one correspondence between the discount function and the time perception: for any discount function ϕ, there is a unique time perception ρ such that $\phi(t) = \lambda^{\rho(t)}$. Therefore, in the SDU model, time discounting results from two distinct sources: a discount factor λ that expresses the initial preference for present and a time perception $\rho(t)$ that determines how impatience changes with time.

Definition 4.3 (Time aggregative function)

For a given time perception ρ, we called time aggregative function the relation F that associates to any pair of delays $\{t', t\}$, such that $\rho(t') + \rho(t) \in \mathcal{P}'$, the quantity:

$$F(t',t) = \rho^{-1}\big(\rho(t') + \rho(t)\big) \tag{4.2}$$

For the initial indifference $(x,0) \sim (y,\tau)$, the time aggregative function $F(t,\tau)$ is the value that maintains indifference when alternatives are delayed by t periods in the future, that is, $(x,t) \sim \big(y, F(t,\tau)\big)$. As a consequence, $F(t,\tau)$ implicitly indicates how the DM subjectively aggregates two delays τ and t. Note that time aggregation has no order since $F(t,t') = F(t',t)$ and is reduced to linear time if one delay is null, that is, $F(0,t) = F(t,0) = t$. Moreover, for linear time perception $\rho(t) = t$, the indifference $(x,t) \sim (y, t+\tau)$ holds and impatience is constant. Thus, τ years from now and τ years starting in t years seem equivalent since $F(t,\tau) = t + \tau$. However, for a logarithmic time perception $\rho(t) = \ln(1+t)$ $(\alpha = 1)$, the indifference $(x,t) \sim (y, t + \tau + \tau t)$ holds and impatience is decreasing. In this latter case, the DM is willing to wait for an additional delay $\tau t > 0$ when alternatives are delayed by t periods. Thus, τ years starting in t years seems shorter than τ years from now and $F(t,\tau) > t + \tau$.The next axiom, called meta-discounting, introduces time perceptions in order to rationalize any DU model.

Axiom A1 (Meta-Discounting)

$\forall x, y, \tau > 0$, and for all delays $t < T$ s.t. $F(\tau, t)$ is well defined,

$$(x,0) \sim (y,\tau) \Longrightarrow (x,t) \sim \big(y, F(\tau,t)\big)$$

In the next section, we will justify that, if the additional delay belongs to the time tradeoff horizon, then the time aggregative function is always defined. Axiom A1 expresses meta-discounting based on a functional parameter ρ. The first indifference defines the DM's discount factor[11]. The second indifference determines the nature of impatience. As a consequence, in the SDU model, the discount factor λ is exogenous whereas the willingness to wait $\tau(t)$ is endogenous and depends on the expression attributed to time perception. The next representation theorem shows that Axiom A1 restricts the expression of DU functions on the set of Subjective Discounted Utility functions.

Theorem 4.4 (Subjective Discounted Utility)

Time preferences $\succsim$ verify (A0) and (A1) if and only if SDU holds.

Axiom A1 has two interpretational keys. The first interpretation holds when a particular form is dedicated to time perception. In this case, Axiom A1 restricts the scope of rationality to a single discounted utility model. Assume that the modeler has in mind some interesting properties attached to a specific discount function. Then, the meta-discounting

[11] Indeed, under DU, $(x,s) \sim (y,l) \Leftrightarrow \phi(l)/\phi(s) = u(x)/u(y) = \lambda$ with $\lambda = 1/(1+r)$, where r is the impatience rate.

axiom associates to each Discounted Utility a corresponding axiom. For instance, if the time perception is linear, then A1 becomes the stationarity axiom $\big((x,0)\sim(y,\tau)\Longrightarrow(x,t)\sim(y,t+\tau)\big)$ and EDU holds. If the time perception is logarithmic, then A1 becomes the specific axiom $\big((x,0)\sim(y,\tau)\Longrightarrow(x,t)\sim(y,t+\tau+\alpha\tau t)\big)$ provided by Loewenstein and Prelec (1992) and HDU holds. In the same way, axiom A1 may be used to modelize discount functions with asymptotic patience such as CADI DU or DADI DU.

Corollary 4.5

Time preferences $\succsim$ are represented by DU. Then,

CADI Discounted Utility holds if and only if time preferences $\succsim$ verify A1 defined for $\rho(t)=1-e^{-ct}$ $(c>$ 0, that is: for all delays $t<T$,

$$(x,0)\sim(y,\tau)\Longrightarrow(x,t)\sim\left(y,t+\tau-\frac{\ln\left(e^{ct}+e^{c\tau}-e^{c(t+\tau)}\right)}{c}\right)$$

DADI Discounted Utility holds if and only if time preferences $\succsim$ verify A1 defined for $\rho(t)=t/(1+at)$ $(a>0)$, that is: for all delays $t<T$,

$$(x,0)\sim(y,\tau)\Longrightarrow(x,t)\sim\left(y,t+\tau+\alpha\tau t\times\left[\frac{(t+\tau)a+2}{1-t\tau a^2}\right]\right)$$

BRW (2009) have provided alternative but equivalent axiom for CADI DU (Definition 5.1)[12]. Corollary 4.5 focuses the study on the case $c>0$ for which time preferences exhibit decreasing impatience. For DADI DU, time perception may be interpreted as a linear time perception psychologically curved by an affine transformation of time. Parameter a may be meaningfully interpreted as a distortion coefficient, that is, the measure of the departure of the DADI discounting function from constant impatience. Indeed, if parameter a tends toward zero, then exponential discounting results. Note that, as parameter a tends toward infinity, the discount factor tends toward one and there is no discounting anymore. In both cases, observe that the time premium is only defined for a restricted set of delays t.

4.2. Psychological foundation for time discounting

The second interpretation associated to Axiom A1 concerns psychological foundations for time discounting and holds when no expression for time perception is mentioned. Then, SDU is consistent with any DU model. Combining time equation (2.1) (section 2) with axiom A1, the willingness to wait may rewritten as $\tau(t)=F(\tau,t)-t$ and measures the subjective time interval for which the DM is still indifferent between the two options.

12 BRW (2009) defined the CADI axiom in the following way. For all $t_1<t_2<t_3$, all $t>0$ and all $x,y,z>0$, $(x,t_1)\sim(y,t_2)$, $(x,t_2)\sim(y,t_3)$ and $(y,t_1+t)\sim(z,t_2+t)$ imply $(y,t_2+t)\sim(z,t_3+t)$. Note that this latter axiom is consistent with any real value c and includes additional cases in comparison with our axiom. For instance, if $t_2=(t_1+t_3)/2$, then the CADI axiom is reduced to stationarity. In the same way, if the CADI axiom is only defined for $t_2<(t_1+t_3)/2$, then the CADI axiom by BRW (2009) is strictly equivalent to our axiom.

Observation 4.6

If time preferences $\succsim$ are represented by SDU and $(x, t_1) \sim (y, t_2)$. Then,

$(*)$ *Asymptotic patience (resp. impatience) holds if and only if $lim_{t\to\infty} \rho(t) = l$ with l a finite number (resp. $lim_{t\to\infty} \rho(t) = \infty$).*

$(**)$ *Decreasing (resp. constant, increasing) holds if and only if the time perception is concave (resp. linear, convex).*

$(***)$ *For any $t < T$, $\theta(t_1, t_2, t) = \rho^{-1}\big(\rho(t_2) + \rho(t_1 + t) - \rho(t_1)\big) - (t_2 + t)$.*

The first assertion of Observation 4.6 $(*)$ provides sound behavioral foundations for strong impatience and for weak impatience. A DM, whose time preferences exhibit strong impatience, has time perception sufficiently sensitive to time to clearly dissociate very distant time intervals with different lengths. Formally, asymptotic impatience implies that, for all $M > 0$ and for all delay t, there exists a time interval t' such that $\rho(t + t') - \rho(t) > M$. On the contrary, a DM whose preferences exhibit weak impatience has time perception insufficiently sensitive to time and very distant time intervals with different lengths will be subjectively perceived with equal length. In particular, asymptotic patience expresses that there exists a delay t such that, for all $t', \varepsilon > 0$, $\rho(t + t') - \rho(t) < \varepsilon$. The two last parts of Observation 4.6 restate earlier results provided by Lapied and Renault (2012). Proposition $(**)$ links the nature of impatience with concavity property of time perceptions. The third proposition $(***)$ explains the time sensitivity of impatience from time perception properties. Rewritten for $t_1 = 0$, the time premium $\theta(0, \tau, t) = F(t, \tau) - (\tau + t)$ may be meaningfully interpreted as the psychological gap between time aggregative function and linear time.

5. Psychological foundations for very distant future

5.1. The timing of time tradeoff horizon

For SDU representation of time preferences, the timing of the time tradeoff horizon T is allowed to drastically vary from a DM to another: T is infinite if time preferences verify asymptotic impatience and T is finite if asymptotic patience holds (Observation 4.6). In this latter case, the time tradeoff horizon may vary from a few weeks to many years. In the SDU model, this variability is fully captured by time perceptions that embody the DM's ability to represent distant time intervals. For instance, consider the dated outcomes $(x, 20\ years)$ and $(y, 21\ years)$ that are both delayed by a minimal twenty year delay. For a DM who is able to clearly represent distant time intervals (one year starting after twenty years seems lengthy), the one year additional delay remains sufficiently costly and time tradeoff holds: in this case, the time tradeoff horizon is very large. On the contrary, for a DM who is not able to clearly represent distant time intervals (one year starting after twenty years seems very short), the one year additional delay is not costly anymore and time tradeoff is fully consumed: in this case, the time tradeoff horizon is brief. The next proposition gives a simple analytical expression to time tradeoff horizon T whose psychological foundations are embodied by the distortions induced by time perceptions.

Proposition 5.1

Time preferences $\succsim$ are represented by SDU. Assume the initial indifference $(x,t_1) \sim (y,t_2)$, then the time tradeoff horizon is:

$$T = \rho^{-1}\big(l - \rho(\tau)\big) \tag{5.1}$$

with $l = lim_{t\to\infty}\rho(t)$ and τ is the initial willingness to wait $(x,0) \sim (y,\tau)$.

Proposition 5.1 gives a complete expression to the SDU model leading to a simpler form for axiom A1. Indeed, by definition 4.3, observe that, for any additional delay $t < \rho^{-1}\big(l - \rho(\tau)\big)$, the time aggregative function $F(\tau,t)$ is well defined. Then, we can rewrite axiom A1 in the following way: $\forall x,y > 0, \forall t_1,t_2 \geq 0$ and $\forall t < T = \rho^{-1}\big(l - \rho(\tau)\big)$, $(x,0) \sim (y,t) \Longrightarrow (x,t) \sim \big(y, F(t,\tau)\big)$.

Furthermore, the time tradeoff horizon is outcome independent[13] and, no matter the periods t_1 and t_2 are, T only depends on τ that reflects the initial preference for present and on the particular shape dedicated to the time perception (including its limit). By observation 2.3, the initial willingness to wait τ exists since if time tradeoff holds in period t_1, then, there is time tradeoff for any delay $t \leq t_1$. By definition 4.1, the inequality $\rho(\tau) < l$ is verified for any $\tau > 0$ and the time tradeoff horizon is well defined. In the proof of Proposition 5.1, we will show that $T > t_1$. When the limit l is infinite (resp. finite), asymptotic impatience (resp. patience) holds and the time tradeoff horizon is infinite (resp. finite). If t_1 is going closer to t_2, then the T value becomes arbitrary large. In this case, the DM is initially highly impatient and time tradeoff may be extended for very long delays. If t_2 becomes arbitrary large, then the T value gets closer to t_1. In this case, the initial indifference is very close to very distant future, so that any additional delay totally consumes time tradeoff opportunity and leads alternatives into very distant future.

Proposition 5.1 clarifies the timing of time tradeoff horizon for all discounted utilities. For standard discounting (exponential and hyperbolic discounting), the time tradeoff horizon is infinite. For DADI DU with $a = 1$ (hence, $l = 1/a = 1$), consider the following indifference $(x,1) \sim (y,2)$. Then, by SDU, $\rho(\tau) = \rho(2) - \rho(1) = 1/6$ and proposition 5.1 implies that $T = 5$. More generally, the time tradeoff horizon may be sharply represented for any delays t_1, t_2.

Corollary 5.2

Suppose that time preferences $\succsim$ are represented by SDU and assume $(x,t_1) \sim (y,t_2)$.

$$If\ CADI\ DU\ holds, then\ T = t_1 - \frac{ln\big(1 - e^{-c(t_2-t_1)}\big)}{c}, with\ \forall t_1 \geq 0, t_1 < T < \infty$$

$$If\ DADI\ DU\ holds, then\ T = \frac{1 + 2at_1 + a^2 t_1 t_2}{a^2(t_2 - t_1)}, with\ \forall t_1 > 0, t_1 < \frac{t_1 t_2}{t_2 - t_1} < T < \infty$$

[13] That is, for any $x', y' > 0$ such that $(x',t_1) \sim (y',t_2)$, the time tradeoff horizon still remains $T = \rho^{-1}\big(l - \rho(\tau)\big)$. This observation is a direct consequence of the separability axiom.

When time preferences are represented by CADI DU, time perception is sufficiently flexible to allow time tradeoff horizon to take any value on $]t_1;\infty[$ depending on the value of parameter c, with $\lim_{c\to 0} T(c) = \infty$ and $\lim_{c\to\infty} T(c) = t_1$. However, when time preferences are represented by DADI DU, the distortion induced by time perception is restricted in the sense that T only can take any value on $](t_1 t_2)/(t_2 - t_1)\,;\infty[$, with $\lim_{a\to 0} T(a) = \infty$ and $\lim_{a\to\infty} T(a) = (t_1 t_2)/(t_2 - t_1)$. Therefore, the psychological distortion cannot eliminate any trace of time tradeoff on the entire time horizon since there exists an incompressible time tradeoff horizon $[0;(t_1 t_2)/(t_2 - t_1)[$. For instance, consider the DADI function defined for any positive parameter a with $(x,1) \sim (y,2)$. Then, the incompressible time tradeoff horizon is $[0;2[$, that is, time tradeoff holds for any delay $t < 2$ no matter the value of parameter a is.

In particular, the existence of incompressible time tradeoff horizons indicates that the index introduced by Prelec (2004) has limited interpretation restricted to "small intervals". The Prelec's index $\gamma(t_1)$ is defined for delays $t_1 > 0$ such that $(x,t_1) \sim (y,t_2)$ and allows comparisons on the decreasing rates of impatience for small time intervals t around the t_1 value. Formally, for two time preferences $\succsim_1$ and $\succsim_2$ respectively represented by $v_1(x,t_1) = \phi_1(t_1) \times u_1(x)$ and $v_2(x,t_1) = \phi_2(t_1) \times u_2(x)$, $\succsim_1$ exhibits more decreasing impatience than $\succsim_2$ if, for twice differentiable discount functions,

$$\gamma_1(t_1) = -\frac{{\phi_1}''(t_1)}{{\phi_1}'(t_1)} + \frac{{\phi_1}'(t_1)}{\phi_1(t_1)} > \gamma_2(t_1) = -\frac{{\phi_2}''(t_1)}{{\phi_2}'(t_1)} + \frac{{\phi_2}'(t_1)}{\phi_2(t_1)}$$

The Prelec's index for hyperbolic discounting is $\gamma_H(t_1) = \alpha/(1 + \alpha t_1)$ and BRW (2009) have shown that, for any positive value of parameter α, the index is bounded, that is, $\gamma_H(t) < {t_1}^{-1}$ (Observation 3.5). BRW (2009) conclude that *"[…] there is an upper bound to the degree of decreasing impatience with decreasing impatience vanishing if t tends to infinity"* (p.35). For SDU representation of time preferences, the Prelec's index is $\gamma(t_1) = -\rho''(t_1)/\rho'(t_1)$ [14] (Lapied and Renault 2012). For DADI functions, it is immediate to show that $\gamma_{DADI}(t_1) = (2a)/(1 + a t_1) = 2 \times \gamma_H(t_1)$ (for $\alpha = a$) and, consequently, the Prelec's index is also bounded: for any delay $t_1 > 0$, $\gamma_{DADI}(t_1) < 2/t_1$. This observation is not inconsistent with the fact that DADI functions exhibit weak impatience, that is, for distant future, the decrease in discount rates is not bounded. Indeed, DADI functions imply the existence of an incompressible time tradeoff horizon and, for any small delay $t \leq (t_1 t_2)/(t_2 - t_1)$, the decrease in the impatience rate is actually bounded. However, for any sufficient distant delay $t > (t_1 t_2)/(t_2 - t_1)$, the decrease in the impatience rate is allowed to be arbitrarily large.

5.2. Additional conditions for time tradeoff

Last, additional conditions for time tradeoff are provided. First, focusing on asymptotic impatience, we identify the set of discount functions for which the decrease in the impatience rate is restricted so that strong decreasing impatience is excluded. In particular, hyperbolic discounting is included into this set. Second, focusing on asymptotic patience, we identify the set of discount functions which are sufficiently flexible to extend time tradeoff for any delay. In particular, this is the case for CADI discounting.

A particular class of discount functions that verifies asymptotic impatience is the set of functions that exclude strong impatience, that is, discount functions for which the time premium $\theta(t_1, t_2, t)$ is bounded by $t \times (t_2 - t_1)/t_1$. BRW (2009) have shown that hyperbolic discounting excludes strong impatience (Section 3). We provide an axiomatic

[14] The twice differentiability of discount function entails the twice differentiability of time perception.

condition, called weak decreasing impatience, which enables us to identify the set of discount functions inconsistent with strong impatience.

Axiome A2 (Weak Decreasing Impatience)

For $0 \leq t_1 < t_2, \forall x, y > 0$ and $\forall m > 1$

$$(x, t_1) \sim (y, t_2) \Longrightarrow (x, mt_1) \succ (y, mt_2)$$

Weak decreasing impatience has meaningful interpretation. Consider the initial indifference $(x, t_1) \sim (y, t_2)$ and assume that both outcomes are delayed by the same amount of time $t = (m-1)t_1$. Axiom A2 says that if the higher outcome is delayed by the additional delay $t^{'} = (m-1)(t_2 - t_1)$, then time preferences turn for the smaller outcome. By definition, axiom A2 excludes asymptotic patience, since, for any $t = (m-1)t_1 > 0$, time tradeoff may occur[15]. As a consequence, weak decreasing impatience necessarily includes discount functions that exhibit constant impatience or increasing impatience. Indeed, for $m = (t_1 + t)/t_1 > 1$, constant impatience or increasing impatience implies $(x, t_1) \sim (y, t_2) \Longrightarrow (x, t_1 + t) \succsim (y, t_2 + t)$. Observe that $mt_2 > t_2 + t$ hence, by impatience and by transitivity, $(x, mt_1) \succ (y, mt_2)$. Moreover, weak decreasing impatience is consistent with discount functions whose impatience rate is slowly declining with time, that is, for which the additional delay $t^{'} = (m-1)(t_2 - t_1)$, so large is it, is sufficient to induce time tradeoff. The next theorem shows that axiom A2 precisely induces the set of discount functions whose time premium is bounded by $t \times (t_2 - t_1)/t_1$.

Theorem 5.3

Assume that time preferences $\succsim$ are represented by SDU. Then SDU excludes strong decreasing impatience if and only if $\succsim$ verify (A2).

Theorem 5.3 generalizes Observation 3.1 by BRW (2009). A large range of DU models satisfy axiom A2. For instance, EDU or CADI DU (for $c \leq 0$) are consistent with A2 since they exhibit constant impatience or increasing impatience. For time preferences that satisfy decreasing impatience, hyperbolic discounting has weak decreasing impatience. In addition, Ebert and Prelec (2007) provided a discount function, $\phi(t) = \lambda^{t^b}$ $(b > 0)$, called power discounted utility by Lapied and Renault (2012), that also verifies axiom A2 for any positive parameter b. Alternatively, we can axiomatize strong decreasing impatience in the following way.

Axiome A3 (Strong Decreasing Impatience)

For $0 \leq t_1 < t_2, \forall x, y > 0, \exists m > 1$ such that

$$(x, t_1) \sim (y, t_2) \Longrightarrow (y, mt_2) \succsim (x, mt_1)$$

Strong decreasing impatience includes weak impatience. The next theorem identifies discount functions for which the time tradeoff equation $(x, t) \sim (y, \tau + t + \theta)$ is always defined, implying maximal flexibility for time tradeoff occurrence.

[15] By continuity and impatience, $\forall t > 0$, $(x, t) \sim \big(y, t + \tau(t)\big)$ with $0 < \tau(t) < t^{'} = (m-1)(t_2 - t_1)$.

Theorem 5.4

Time preferences are represented by SDU and consider the following indifference $(x, 0) \sim (y, \tau)$*. Then, the time tradeoff equation* $(x, t) \sim (y, \tau + t + \theta)$ *is defined for all* $t > 0$ *and all* $\theta > 0$ *if and only if the time perception* $\rho(\tau)$ *can take any value on* $]0; l[$ *(l finite) for any delay* τ*.*

Theorem 5.4 generalizes Observation 7.1 by BRW (2009) stressing that the time tradeoff equation is defined for all $t > 0$ and all $\theta > 0$ if CADI discounting holds. For DU models consistent with asymptotic impatience, the time tradeoff equation (2.2) is only defined for restricted values of θ. Then, Theorem 5.4 focuses the analysis on discount functions satisfying asymptotic patience and restricts the representation of time preferences in two distinct ways. First, time perceptions are sufficiently flexible to extend time tradeoff to very large delays since, if $\rho(\tau)$tends toward zero, the time tradeoff horizon becomes arbitrary large. Second, time perceptions are sufficiently flexible to consume time tradeoff for very small delays since, if $\rho(\tau)$tends toward l, the time tradeoff horizon becomes null. CADI discounting is a good candidate since $\lim_{c \to 0} \rho(\tau) = 0$ and $\lim_{c \to \infty} \rho(\tau) = 1$. On the contrary, DADI discounting is not sufficiently flexible since it exhibits incompressible time tradeoff horizon on which the time premium is bounded. Last, note that the discount function $\phi(t) = \lambda^{1-e^{-c\rho_1(t)}}$ (the function $\rho_1(t)$ is a time perception such that $\lim_{t \to \infty} \rho_1(t) = \infty$) also satisfies the condition stressed by Theorem 5.4.

6. Conclusion

In this paper, an axiomatic condition, called asymptotic patience, is defined to characterize very distant future, that is, a time horizon on which there is no time tradeoff. However, standard Discounted Utilities imply time tradeoff and there is no very distant future. A new DADI discounting rule is provided, both consistent with time tradeoff for initial delays and with asymptotic patience for more distant delays. In this case, individual time preferences defined on very distant future exhibit a zero discount rate. This useful property may be used to describe how individual decision makers care about welfare in situations such as global warming, when main impacts will occur only in a few centuries. In particular, the well-known Ramsey's rule, which partially depends on the constant discount rate r, could be generalized by an alternative Ramsey's rule, where the discount rate r_t is not constant anymore and falls to zero for very long delays.

Appendix. Proofs

Proof of Observation 2.3

(i) For any dated outcome (y, τ) $(y > 0)$, impatience implies $(y, 0) \succ (y, \tau)$ and, since $u(0) = 0$, $(y, \tau) \succ (0,0)$. Then, by continuity, there exists a unique outcome x $(0 < x < y)$ such that $(x, 0) \sim (y, \tau)$, where x is called present equivalent of dated outcome (y, t). If impatience is constant, then, $\forall t > 0$, $\tau(t) = \tau$ and $(x, t) \sim (y, t + \tau)$. Assume now increasing impatience holds, then $\forall t > 0$, $(x, t) \succ (y, t + \tau)$ and, by impatience and by monotonicity, there is a unique $\tau(t)$ $(\tau(t) < \tau)$ such that $(x, t) \sim \big(y, t + \tau(t)\big)$.

(ii) Assume that, for a given delay $t_2 > 0$, there is time tradeoff, that is $(x, t_2) \sim \big(y, t_2 + \tau(t_2)\big)$. Then, for $t_1 < t_2$, monotonicity implies $(y, t_1) \succ (x, t_1)$ and impatience implies $(x, t_1) \succ (x, t_2)$. By transitivity, $(y, t_1) \succ (x, t_1) \succ \big(y, t_2 + \tau(t_2)\big)$ and, by continuity and impatience, there is a unique $\tau(t_1)$ $(0 < \tau(t_1) < \tau(t_2) + t_2 - t_1)$ such that $(x, t_1) \sim \big(y, t_1 + \tau(t_1)\big)$.

(iii) If there is no time tradeoff for delay t_1 then, for any $t > 0$, $(y, t_1 + t) \succ (x, t_1)$. By decreasing impatience, for all $t_2 > t_1$, we have $(y, t_2 + t) \succ (x, t_2)$.

Proof of Observation 3.3

The proof of the first assertion of Observation 3.3 is immediate since, by Observation 2.3 (i), if impatience is constant or increasing with time, then there is time tradeoff for any delay t.

Assume now that, from a delay t, constant impatience or increasing impatience holds. Then, by observation 2.3, there is time tradeoff for any delay $t^{'} > t$, a contradiction with weak impatience.

Proof of Proposition 3.6

$(ii) \Leftrightarrow (iii)$ If $lim_{t\to\infty} \phi(t) = 0$, then, since ϕ is continuous and strictly decreasing on $\mathcal{P}$, for any $t_1 \in \mathcal{P}$ and $\varepsilon > 0$, there is $t_2 > t_1$ such that $\phi(t_2)/\phi(t_1) < \varepsilon$. Conversely, assume that $\lim_{t\to\infty} \phi(t) = L > 0$ and, for a real number m such that $1/L > m > 1$, set $\phi(t) = mL$. Thus, for $\varepsilon = 1/m > 0$, $\forall t^{'} > t$, $\phi(t^{'})/\phi(t) > \varepsilon$, a contradiction with the squashing property.

$(i) \Leftrightarrow (iii)$ Monotonicity implies that, for any $t > 0$, $(y, t) \succ (x, t)$ and, by DU, $u(x)/u(y) < 1$. Assume that $\lim_{t^{'}\to\infty} \phi(t + t^{'}) = 0$. Thus, by continuity of the discount function, $\forall t^{'} > 0$, $\phi(t + t^{'})/\phi(t) \in]0; 1[$. Consequently, $\exists t^* > 0$ such that $u(x)/u(y) > \phi(t + t^*)/\phi(t)$ and, by DU, $(x, t) \succ (y, t + t^*)$. Assumption A0 finally implies that there exists $\tau(t)$ $(0 < \tau(t) < t^*)$ such that $(x, t) \sim \big(y, t + \tau(t)\big)$. Reciprocally, assume that $\lim_{t\to\infty} \phi(t) = L > 0$. Choose $t > 0$ such that $\phi(t) = L/m$ $(L < m < 1)$, thus, $\forall t^{'} > 0$, $\phi(t + t^{'})/\phi(t) > m$. Therefore, for any pair of outcomes $x, y > 0$ such that $u(x)/u(y) = m^- < m$, the inequality $u(x)/u(y) < \phi(t + t^{'})/\phi(t)$ results, a contradiction with time tradeoff.

Proof of Theorem 4.4

Observe first that if $\rho(t) + \rho(\tau) < l$ then the time aggregative function is well defined and there is time tradeoff for period t. By Lapied and Renault (2012) (Theorem 2), we deduce the SDU representation for time preferences.

Proof of Corollary 4.5

The proof of corollary 4.5 is directly deduced from Theorem 4.4. For DADI functions, $\theta = \alpha\tau t \times \big((t + \tau)a + 2\big)/(1 - t\tau a^2)$. Then, the time premium associated to DADI functions departs from the time premium linked to hyperbolic functions $(\theta = \alpha\tau t)$ by an extra factor $\big((t + \tau)a + 2\big)/(1 - t\tau a^2)$. Moreover, note that, in both cases, time premiums are defined for a restricted set of additional delay t that reflects that time tradeoff cannot be extended for very distant future. For CADI axiom, θ is well defined if $t < \tau - \ln(e^{c\tau} - 1)/c$ and $e^{ct}(1 - e^{c\tau}) < 1 - e^{c\tau}$ implies that $\theta = -\ln\big(e^{ct} + e^{c\tau} - e^{c(t+\tau)}\big)/c > 0$. For DADI functions, θ is well defined if $t < 1/(a^2\tau)$, hence $\theta = \alpha\tau t \times \big((t + \tau)a + 2\big)/(1 - t\tau a^2) > 0$.

Proof of Observation 4.6

The proof of assertion $(*)$ results from Proposition 3.3 and Theorem 4.4. The proof of assertion $(**)$ is given by Lapied and Renault (2012) (Theorem 3). Last, assume time tradeoff for t_1 and $t_1 + t$, that is $(x, t_1) \sim (y, t_2)$ and $(x, t_1 + t) \sim \big(y, t_2 + t + \theta(t_1, t_2, t)\big)$. By SDU, we have $\rho(t_2) - \rho(t_1) = \rho(t_2 + t + \theta) - \rho(t_1 + t)$ and assertion $(***)$ holds.

Proof of Proposition 5.1

Assume $(x, t_1) \sim (y, t_2)$, then, by Observation 2.3, there exists a delay τ such that $(x, 0) \sim (y, \tau)$. By SDU, we have $\rho(t_2) - \rho(t_1) = \rho(\tau)$. Moreover, by Observation 4.6, there is time tradeoff if $\rho(t_2) + \rho(t_1 + t) - \rho(t_1) < l$, that is, if $t_1 + t < \rho^{-1}(l + \rho(t_1) - \rho(t_2)) = \rho^{-1}(l - \rho(\tau))$. The delay $T = \rho^{-1}(l - \rho(\tau))$ indicates the time tradeoff horizon and the delay $T - t_1 = \rho^{-1}(l - \rho(\tau)) - t_1$ indicates the set of additional delays t for which time tradeoff in period t_1 may be extended.

Note that :

(1) $\rho(t_1) - \rho(t_2) < 0 \Rightarrow l + \rho(t_1) - \rho(t_2) < l$

(2) $l - \rho(t_2) > 0 \Rightarrow l + \rho(t_1) - \rho(t_2) > 0$

(3) $l - \rho(t_2) > 0 \Rightarrow \rho^{-1}[l + \rho(t_1) - \rho(t_2)] > t_1 \Rightarrow T > t_1$

(4) $lim_{l\to\infty} T(l) = +\infty, lim_{l\to\rho(t_2)}[T(t_2) - t_1] = 0, lim_{t_2\to\infty}[T(t_2) - t_1] = 0, lim_{t_1\to t_2}[T(t_2) - t_1] = +\infty$

Conditions (1) and (2) imply that the time tradeoff horizon T is well defined and (3) implies that T is higher than t_1 since we assume time tradeoff for period t_1. Moreover, (4) implies that, as l tends toward infinity, the choice structure switches to the asymptotical impatience case and, as l tends toward $\rho(t_2)$, the first indifference is very closed to very distant future. In addition, the smaller (resp. larger) the difference between t_1 and t_2 is and the larger (resp. smaller) the time tradeoff horizon is.

Proof of Corollary 5.2

By proposition 5.1, the proof results from straightforward algebraic manipulations not given here.

Proof of Theorem 5.3

For SDU representation of time preferences, A2 is equivalent to $\rho(ml) - \rho(ms) > \rho(l) - \rho(s)$ $(m > 1)$, which defines a particular class of time perceptions. We show that this class gathers precisely the set of time perceptions that exclude strongly decreasing impatience. Assume A2 holds, then, by definition of time perception, $s < l$ implies $\rho(l) - \rho(s) > 0$. For all $k > 0$, let us multiply the two periods both by $m = (s + k)/s > 1$, so $\rho(l\,(s + k)/s) - \rho(s + k) > 0$. Then A2 implies $\rho(l) - \rho(s) < \rho(l\,(s + k)/s) - \rho(s + k)$, that is $\rho(l) - \rho(s) + \rho(s + k) < \rho(l\,(s + k)/s)$. By definition of time perception again, $\rho^{-1}(\rho(l) - \rho(s) + \rho(s + k)) < l\,(s + k)/s$ and, by definition of time premium, $\theta(s, l, k) = \rho^{-1}(\rho(l) - \rho(s) + \rho(s + k)) - (l + k)$. As a result, $\theta(s, l, k) + (l + k) < l\,(s + k)/s$, that is $\theta(s, l, k) < k\,(l - s)/s$. Reciprocally, assume $\theta < k\,(l - s)/s$ and let us show that A3 holds. Assume $(x, s) \sim (y, l)$, then, for all period $k < K$, SDU implies $\rho(l + k + \theta) - \rho(s + k) = \rho(l) - \rho(s)$. By hypothesis, $\theta < k\,(l - s)/s$ and, by definition of time perception, $\rho(l + k + k\,(l - s)/s) - \rho(s + k) > \rho(l) - \rho(s)$, that is $\rho(l + l \times k/s) - \rho(s + k) > \rho(l) - \rho(s)$. Setting $k = (m - 1)s$ $(m > 1)$, $\rho(ml) - \rho(ms) > \rho(l) - \rho(s)$ and A2 results.

Proof of Theorem 5.4

First, note that if asymptotic impatience holds (l is infinite), then the time premium is bounded. As a consequence, we will focus on the case of asymptotic patience (l is finite). By Proposition 5.1, $T = \rho^{-1}(l - \rho(\tau))$. By definition 4.1, ρ^{-1} is continuous and strictly increasing on $]0; l[$. Thus, T is continuous and strictly decreasing with $\rho(\tau) \in]0; l[$.

First, assume that $\rho(\tau)$ can take any value on $]0;l[$. If $\rho(\tau)$ tends toward zero, then the time tradeoff horizon T tends toward infinity and if $\rho(\tau)$ tends toward the l value, then the time tradeoff horizon T tends toward zero. Continuity and strict monotonicity of time tradeoff horizon imply that T can take any value on $]0;\infty[$. As a consequence, for any additional delay $t > 0$, we can define T such that $t < T$ and the time tradeoff equation is well defined. Moreover, for a given additional delay t, there exists a unique $\mu^* \in]0;l[$ such that, when $\rho(\tau)$ tends toward μ^*, the time tradeoff horizon T tends towards t. Then, for $\rho(\tau) \in]0;\mu^*[$, we have $T \in]t;\infty[$.

Lemma 1: $\forall\tau > 0, \lim_{t\to T} \theta(0,\tau,t) = \infty$

Proof of Lemma 1. Weak impatience implies that $\rho : \mathcal{P} \to [0;l[$. By Proposition 5.1, $T = \rho^{-1}\big(l - \rho(\tau)\big)$ and, by Observation 4.6, $\theta(0,\tau,t) = \rho^{-1}\big(\rho(\tau) + \rho(t)\big) - (\tau + t)$, so $\lim_{t\to T} \theta(0,\tau,t) = \lim_{t\to l} \rho^{-1}(t) = \infty$.

By lemma 1, the time premium $\theta(0,\tau,t)$ becomes infinite as $\rho(\tau)$ is going closer to μ^*. In addition, by observation 4.6, $\lim_{\rho(\tau)\to 0} \theta(0,\tau,t) = -\tau$. Note that Lemma 3.2 by BRW (2009) indicates that DU assumptions imply $\theta > -\tau$ and $\theta > -t$ and observe that, if $t < \tau$, that is $\rho(t) < \rho(\tau)$, then $\rho(\tau) \to 0$ implies $\rho(t) \to 0$ and $\lim_{\rho(t)\to 0} \theta(0,\tau,t) = -t$. Then, there exists a unique $\mu_* \in]0;\mu^*[$ such that, when $\rho(t)$ tends toward μ_*, the time premium θ tends toward zero. Consequently, for $\rho(\tau) \in]\mu_*;\mu^*[$, we have $\theta \in]0;\infty[$.

Reciprocally, assume that the time perception can take only a restricted set of values on $]0;l[$. For instance, consider first $\rho(\tau) \in]t^-;l[$ or $\rho(\tau) \in]0;t^+[$ with $0 < t^- < t^+ < l$, then $\rho^{-1}(l - t^-) > T > \rho^{-1}(l - t^+)$. As a consequence, there is an incompressible time tradeoff horizon $\big(T > \rho^{-1}(l - t^+)\big)$ and, for any additional $t > \rho^{-1}(l - t^-)$, the time tradeoff equation is not defined. Second, remark that the case $\rho(\tau) \in]0;l[- [t^-;t^+]$, with $0 < t^- \leq t^+ < l$, is reduced to the case of $\rho(\tau) \in]t^-;l[$ since, by definition, if time tradeoff horizon is not defined for $\rho(\tau) = t^-$, then time tradeoff horizon is not defined for any t' such that $\rho(\tau) = t' < t^-$.

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