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# WORKING PAPER

## An equilibrium model of city with atmospheric pollution dispersion

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We propose a spatial model of city coupling a labour market, a residential market, and air pollution resulting from commuter traffic. The city can be of any shape. Agents choose where to work and live in order to maximize their utility, by consuming goods, residential surface and by valuing air quality. Pollution dispersion is described by a scalar transport equation, accounting for meteorological effects (diffusion, advection by wind, and lessivage by rain). We prove existence of equilibria, and we propose an algorithm for computing solutions. We obtain analytical and numerical results emphasizing the combined role of economic and meteorological factors on urban air quality. We finally address welfare considerations.

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# Executive summary

Urban air pollution is a co multifaceted problem that requires a comprehensive and interdisciplinary approach to understand and address. Atmospheric dispersion models are powerful tools that are widely used to simulate and predict the dispersion and concentration of pollutants in the air. However, these models alone are not able to capture the human activities that drive pollution emissions. On the other hand, urban economic models can provide valuable insights into the economic activities and land use patterns that contribute to pollution.

The connection between these two complementary approaches is noteworthy, yet still much unexplored. The urban economics literature that has so far focused on air pollution has quite ignored the phenomenon of atmospheric dispersion. This theoretical work develops a model of city coupling a labour market, a residential market and pollution resulting from commuter traffic. We use a law of motion for pollution that accounts for meteorological effects (for example, transport by wind, or lessivage by rain) and that is inspired by atmospheric modelling.

Further research in this direction would make it possible to predict more accurately the impact of urban policies, whether economic or environmental, on the urban activity and air quality.

## 1 INTRODUCTION

Urban air pollution is a complex and multifaceted problem that requires a comprehensive and interdisciplinary approach to understand and address. Atmospheric dispersion models are powerful tools that are widely used to simulate and predict the dispersion and concentration of pollutants in the air. However, these models alone are not able to capture the human activities that drive pollution emissions. On the other hand, urban economic models can provide valuable insights into the economic activities and land use patterns that contribute to pollution.

The connection between these two complementary approaches is noteworthy, yet still very unexplored. The urban economics literature that has so far focused on endogenous air pollution (for example, [Arnott et al., 2008], [Schindler et al., 2017], [Regnier and Legras, 2018], [Kyriakopoulou and Picard, 2021]) has largely ignored the phenomenon of atmospheric dispersion. Furthermore, it has relied on stylized spatial settings, often assuming the city is linear and monocentric. According to [Wegener, 2019], "*Today only few urban models are linked to environmental models to show the impact of planning policies on greenhouse gas emissions, air quality, traffic noise and open space.* [...] *Even fewer models are able to model the reverse relationship, the impact of environmental quality, such as air quality or traffic noise, on location*".

In this paper, we propose a unifying framework, based on a model developed in [Achdou et al., 2022] and, more extensively, in [Petit, 2022]. We consider a closed, plane city of any shape, in which there is a continuum of workers and firms. First, individuals can freely choose where they work and live. They aim at maximizing their utility, by consuming goods, housing surface and by valuing air quality. Second, firms are distributed continuously throughout the city, allowing to model one or several business districts, located anywhere. Third, pollution arises from automobile commuting, and its dispersion is described through an advection-diffusion equation, allowing to account for meteorological effects such as diffusion, transport by wind and lessivage by rain. The source term of this equation corresponds to the traffic flow. It thus depends on where people work and live, and makes the coupling with the housing and labour markets.

We use a fixed-point argument to prove existence of equilibria, in wages, rental prices, population density and pollution. We also propose a numerical method for computing solutions.

We then examine some characteristics of the model by analyzing the impact of pollution aversion and wind on the equilibrium. To discuss the effect of pollution aversion, we use the linear monocentric city example. We derive an analytical expression for the equilibrium in this case and find that as residents become more averse to pollution, they tend to concentrate in suburban areas, leading to an increase in overall pollution levels. This result aligns with previous theoretical works such as [Schindler and Caruso, 2022]. We support it with 2D numerical simulations.

Regarding the role of wind, we show that the level of pollution experienced by residents is determined by both economic and meteorological factors, specifically the relative direction of wind and the revenue gradient. If the wind and revenue gradient are oriented in the same direction, pollution is carried to high-revenue areas, where residents concentrate, resulting in an increase in experienced pollution levels. The reverse conclusion holds if they are oriented in opposite directions.

We finally perform a welfare analysis of our model. We show that the labour market is efficient in the sense that the clearing wage maximizes its total surplus. On the other hand, the residential market is inefficient: the clearing rental price does not maximize overall residents' utility, because the agents do not internalize the effects of their location decisions on air pollution.

Our model is quite robust. We make standard assumptions on agents' utility function, commuting cost and firms' demand for labour. For the sake of simplicity, we assume car commuting as being the only source of pollution, but other sources, such as residential heating or industrial processes, could easily be included to the model without changing the core of the demonstration.

This work relates to a recent strand in the literature that has started to explore the role of pollution dispersion in spatial economics problems. [Camacho and Pérez-Barahona, 2015] examine the problem of optimal land use with pollution diffusion. [Boucekkine et al., 2022] study a differential game of investment and depollution with transboudary pollution. As far as we know, our work is the first to address this topic in the case of urban economics.

Some questions remain open. Proving uniqueness is a difficult exercise and is left for future research. Economically, we ignore the positive externality effect of production (the concentration of employment at a given place increases firms' productivity), which is yet central to the very existence of cities ([Fujita and Ogawa, 1982], [Lucas and Rossi-Hansberg, 2002]). Furthermore, we assume that the distribution of firms is exogenous. Finally, our description of pollution dispersion can still be improved: for example, we ignore turbulent effects, which have yet an important role in the dispersion process in urban areas, characterized by complex geometries ([Bahlali et al., 2019]).

As an opening remark, we would like to point out the long-term socioeconomic consequences of pollution advection by wind. For example, it is known that westerly winds in the 19th century contributed to making eastern neighborhoods of Western European capitals more polluted and deprived than their western counterparts ([Heblich et al., 2021]). In some cities, such as Paris or London, this spatial inequality has persisted to our time. A possible extension of our model would be to account for heterogenous agents, in order to capture this spatial inequality.

The paper is organized as follows. In Section 2, we describe the model and define the equilibria. We prove existence of equilibria in Section 3. Section 4 presents some analytical properties of our model. Section 5 is dedicated to the numerical simulations. Section 6 concludes and indicates directions for future research.

## 2 MODEL

We represent the city as an open, convex and bounded subset $\Omega$ of $\mathbb{R}^2$ with smooth boundary. There is a continuum of rational resident-workers (or agents). They supply labour and receive wages from competitive firms that produce a unique numéraire good that is both consumed within the city and sold to the larger economy at the same normalized price. Firms are assumed to be immobile and are heterogeneously concentrated across the city. On the opposite, agents can freely choose where they work and live.

### 2.1 Agents utility and revenue

*Utility.* Given a revenue $R$, a rental price by surface unit $Q$, and a pollution level $\tilde{E}$, the indirect utility function of a generic agent is assumed to be

$$U_{\theta,\gamma}(R,Q,\tilde{E}) = \sup_{C,S} \{C^{\theta}S^{1-\theta}\tilde{E}^{-\gamma},\ C + QS \leq R, C \geq 0, S \geq 0\}$$

where $\theta \in [0,1]$ is the preference for consumption, $\gamma \in [0,+\infty)$ is the aversion to pollution exposure, $C$ denotes the level of consumption of the agent, and $S$ the surface of the residence. This utility function is standard and can be found, for example, in [Schindler et al., 2017] and [Borck, 2019].

Applying the first-order conditions gives, for any $(R,Q) \in (0,+\infty)^2$, the optimal consumption and demand for surface, as

$$C_{\theta}(R) = \theta R \tag{1}$$

and

$$S_{\theta}(R,Q) = (1-\theta)\frac{R}{Q} \tag{2}$$

For any $(R, Q, \tilde{E}) \in (0, +\infty)^3$, the utility of an agent is therefore given by

$$U_{\theta,\gamma}(R, Q, \tilde{E}) = \theta^\theta (1-\theta)^{1-\theta} \frac{R}{Q^{1-\theta} \tilde{E}^\gamma} \tag{3}$$

*Revenue.* We assume that given a wage map $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$, agents at the position $x \in \overline{\Omega}$ and working at $y \in \overline{\Omega}$ receive the income $w(y) - c(x, y)$. The map $c \in C(\overline{\Omega}^2, \mathbb{R}_+)$ represents the commuting cost from $x$ to $y$. To maximize their utility given by (3), agents living at $x \in \overline{\Omega}$ will choose a workplace $y \in \overline{\Omega}$ that maximizes their revenue $w(\cdot) - c(x, \cdot)$. Therefore, the revenue of an agent is given by

$$R(x, w) = \max_{y \in \overline{\Omega}} \{w(y) - c(x, y)\}$$

This maximum can be approximated by a regularized revenue

$$R_\sigma(x, w) = \sigma \ln \left( \int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}} \mathrm{d}y \right) \tag{4}$$

where $\sigma > 0$ is a small regularizing parameter. This approximation, very common in quantitative urban models, can also be interpreted as a representation of idiosyncratic preferences among workers ([Diamond, 2016], [Achdou et al., 2022]). It implies that the probability density for an agent located at $x \in \overline{\Omega}$ to choose the workplace $y \in \overline{\Omega}$ follows a Gibbs distribution:

$$G_\sigma(x, y, w) = \frac{e^{\frac{w(y)-c(x,y)}{\sigma}}}{\int_\Omega e^{\frac{w(z)-c(x,z)}{\sigma}} \mathrm{d}z} \tag{5}$$

## 2.2 Labour market

Let $\mathcal{P}_c(\overline{\Omega})$ be the set of probability measures on $\overline{\Omega}$ that admit a continuous density with respect to the Lebesgue measure. For any distribution of residents $\mu \in \mathcal{P}_c(\overline{\Omega})$ and any wage function $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$, the density of labour supply in $y \in \overline{\Omega}$ is given by

$$\int_{\overline{\Omega}} G_\sigma(x, y, w) d\mu(x)$$

where $G_\sigma(x, y, w)$ is given by (5). By the law of total probability, this is simply the integral, over all the living places $x \in \overline{\Omega}$, of the density of residents at $x$, multiplied by the probability density for an agent to work at $y$ knowing that she resides at $x$.

We assume that the demand for labour at $y \in \overline{\Omega}$, where the wage level is $v > 0$, is given by $L(y, v)$, where $L \in C(\overline{\Omega} \times \mathbb{R}_+^*, \mathbb{R}_+)$. The dependence in $y$ captures the heterogenous concentration of firms across space. The labour market clearing condition thus writes, for every $y \in \overline{\Omega}$:

$$\int_{\overline{\Omega}} G_\sigma(x, y, w) d\mu(x) = L(y, w(y)) \tag{6}$$

## 2.3 Housing market

Now, for any distribution of residents $\mu \in \mathcal{P}_c(\overline{\Omega})$, any wage function $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ and any rental price function $Q \in C(\overline{\Omega}, \mathbb{R}_+)$, the demand for surface is given, for all $x \in \overline{\Omega}$, by

$$S_\theta(R_\sigma(x, w), Q(x))\mu(x)$$

It is the individual demand for surface, given by (2), multiplied by the density of residents at $x$.

We assume that the housing supply is exogenous and equal to 1.[1] The housing market clearing condition then writes, for all $x \in \overline{\Omega}$:

$$S_\theta(R_\sigma(x,w), Q(x))\mu(x) = 1 \tag{7}$$

### 2.4 Pollution dispersion

We assume that the pollution concentration $\tilde{E}(z)$ at $z \in \overline{\Omega}$ can be decomposed into two terms: a background regional level $E_0 > 0$ and a local level $E(z)$, such that $\tilde{E}(z) = E_0 + E(z)$.[2]

The dispersion of local pollution involves several physical and chemical processes, the main ones being:

- *Advection*, which refers to the transport of pollution by wind;
- *Molecular diffusion*, which reflects that pollution naturally spreads from high concentration to low concentration areas;
- *Chemical interactions* between the emitted pollutants and chemical species in the air;
- *Lessivage*, which is the process of natural air purification (for example, by rain).

According to [Sportisse, 2009], the stationary distribution of $E$ solves the following scalar transport equation:

$$\underbrace{\mathbf{V}(z)\cdot\nabla E(z)}_{advection} = \underbrace{\nabla\cdot(k\nabla E(z))}_{diffusion} + \underbrace{\chi(E(z);z)}_{chemical\ interactions} + \underbrace{f(z)}_{source\ term} - \underbrace{\lambda E(z)}_{lessivage},\ \forall z \in \Omega \tag{8}$$

where $\mathbf{V}(z) \in \mathbb{R}^2$ is the wind field at $z \in \Omega$, $k \in (0,+\infty)$ the diffusion coefficient, $\chi : [0;\infty)\times\Omega \longrightarrow \mathbb{R}$ is the chemical interactions function, and $\lambda \in (0,+\infty)$ the lessivage coefficient.

The wind field $\mathbf{V}$, together with a pressure field $p$, typically solves the the two-dimensional stationary Navier-Stokes equations

$$\begin{cases} -\Delta\mathbf{V}(z) + R(z)(\mathbf{V}(z)\cdot\nabla)\mathbf{V}(z) + \nabla p(z) = 0,\ \forall z \in \Omega \\ \nabla\cdot\mathbf{V}(z) = 0,\ \forall z \in \Omega \\ \mathbf{V}(s) = \xi,\ \forall s \in \partial\Omega \end{cases} \tag{9}$$

where $\xi \in \mathbb{R}^2$ and $R \in C(\Omega, \mathbb{R}_+)$ are given. The first equation comes from the law of conservation of momentum, and the second one, also called "continuity equation", from the law of mass conservation. The third equation is a Dirichlet boundary condition. In the sequel, we will only use the continuity equation $\nabla\cdot\mathbf{V} = 0$ on $\Omega$, and the boundary condition $\mathbf{V} = \xi$ on $\partial\Omega$.

We neglect in equation (8) the chemical interactions and assume, without loss of generality, that $k = 1$. This equation then becomes:

$$\Delta E(z) - \mathbf{V}(z)\cdot\nabla E(z) - \lambda E(z) + f(z) = 0.$$

The only thing left is to clarify the source term $f(z)$. The only source of emissions we consider in this paper is car commuting. We ignore other sources, such as residential heating or industrial processes. These sources are important in real-world scenarios, but are less relevant for the theory presented in this paper. We assume that the commuting path is a straight line from home to work. The road network is very dense and can be viewed as a continuum. It has a certain width $\delta > 0$. Then, for any $z \in \Omega$, the flux of individuals commuting by the element of road at $z$ is

$$f_{\mu,w}(z) = \int_{\overline{\Omega}^2} \delta^{-1}\mathbf{1}_{z\in\Sigma_\delta(x,y)}\mu(x)G_\sigma(x,y,w)dxdy \tag{10}$$

[1] The results of this paper still hold if we consider a location dependent supply $\varphi \in C(\overline{\Omega}, \mathbb{R}_+)$ or an isoelastic supply $\psi(Q) = Q^\rho$ with $\rho > 0$.

[2] Background pollution originates at a larger scale and is independent from local emissions ([Tchepel et al., 2010]).

where $\Sigma_\delta(x,y) := \{s \in \Omega, |t-s| \leq \delta, \ \forall t \in [x;y]\}$ is the surface of the rectangle of length $|x-y|$, of width $\delta$, centered around the segment $[x;y]$. Note that for all $(\mu,w) \in \mathcal{P}_c(\overline{\Omega}) \times C(\overline{\Omega}, \mathbb{R}_+^*)$, $f_{\mu,w}$ is measurable, belongs to $L^\infty(\Omega)$ and $\|f_{\mu,w}\|_{L^\infty} \leq \delta^{-1}$.

The previous equation then writes:

$$\Delta E(z) - \mathbf{V}(z) \cdot \nabla E(z) - \lambda E(z) + f_{\mu,w}(z) = 0$$

The value of local pollution at the boundary is supposed to be zero: the borders of the city correspond to rural areas with little pollution. Thus, for all $s \in \partial\Omega$, $E(s) = 0$.

The equation for local pollution dispersion finally takes the following form

$$\begin{cases} \Delta E(z) - \mathbf{V}(z) \cdot \nabla E(z) - \lambda E(z) + f_{\mu,w}(z) = 0, \ \forall z \in \Omega \\ E(s) = 0, \ \forall s \in \partial\Omega \end{cases} \tag{11}$$

We will consider weak solutions to equation (11), as defined in the following. We denote by $H_0^1(\Omega)$ the first order Sobolev space on $\Omega$ with zero boundary value.

Definition 2.1. *We say that $u \in H_0^1(\Omega)$ is a weak solution to (11) if for all $v \in H_0^1(\Omega)$,*

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\mathbf{V} \cdot \nabla u)\, v + \lambda \int_\Omega uv = \int_\Omega f_{\mu,w} v$$

## 2.5 Equilibrium

We define an equilibrium as follows.

Definition 2.2. *We say that $(w(\cdot), Q(\cdot), E(\cdot), \mu) \in C(\overline{\Omega}, \mathbb{R}_+^*)^2 \times \left(H_0^1(\Omega) \cap C(\overline{\Omega}, \mathbb{R}_+^*)\right) \times \mathcal{P}_c(\overline{\Omega})$ is an equilibrium if*

$$\int_{\overline{\Omega}} G_\sigma(x,y,w) d\mu(x) = L(y, w(y)), \ \forall y \in \overline{\Omega} \tag{12}$$

$$S_\theta(R_\sigma(x,w), Q(x))\mu(x) = 1, \ \forall x \in \overline{\Omega} \tag{13}$$

$$-\Delta E(z) + \mathbf{V}(z) \cdot \nabla E(z) + \lambda E(z) = f_{\mu,w}(z), \ \forall z \in \Omega \ ; \ E(s) = 0, \ \forall s \in \partial\Omega \tag{14}$$

$$\operatorname{supp} \mu \subset \operatorname*{argmax}_{x \in \overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q(x), E(x)) \tag{15}$$

In Definition 2.2, equation (12) reflects the equilibrium in the labour market, (13) the one in the housing market, (14) the dispersion of pollution, and (15) is a mobility condition: residents choose to locate at places that maximize their utility. This condition implies that at the equilibrium, all the agents get the same utility level. It can also be seen as a Nash equilibrium condition. Indeed, if $\overline{\Omega} \ni x \mapsto U_{\theta,\gamma}(R_\sigma(x,w), Q(x), \tilde{E}(x))$ is continuous, then (15) is equivalent to

$$\int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q(x), \tilde{E}(x)) d\mu(x) = \sup_{\nu \in \mathcal{P}_c(\overline{\Omega})} \int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q(x), \tilde{E}(x)) d\nu(x) \tag{16}$$

which is a mean-field equation characterizing a Nash equilibrium with a continuum of players.[3]

[3] We refer to [Petit, 2022], section 1.2.2, for a short introduction to static mean field games.

# 3 EXISTENCE

## 3.1 Standing assumptions and main result

We list here our standing assumptions, on the labour demand and transport cost functions.

Assumption 3.1. *The labour demand function $L$ is differentiable and has separated variables, i.e. there exists $\nu \in C^1(\Omega, \mathbb{R}_+)$, $\ell \in C^1(\mathbb{R}_+^*, \mathbb{R}_+)$ such that for all $(y, v) \in \overline{\Omega} \times \mathbb{R}_+^*$, $L(y, v) = \nu(y)\ell(v)$. Besides, $1/\eta \leq \nu \leq \eta$ for some $\eta > 0$. Finally, $\ell$ is decreasing and such that $\lim_{v \to 0^+} \ell(v) = +\infty$ and $\lim_{v \to +\infty} \ell(v) = 0$.*

An example of labour demand function satisfying the conditions of Assumption 3.1 is given below.

*Example* 1. Let $f : \mathbb{R}_+ \to$ be a differentiable production function satisfying the usual Inada conditions.[4] Define the profit of a firm as

$$\pi(w) := \sup_{l \geq 0} \{f(l) - lw\}, \tag{17}$$

where $l$ represents the quantity of labour and $w$ the wage.

The labour demand of an individual firm is

$$l^*(w) = f'^{-1}(w) \tag{18}$$

Let $\nu : \overline{\Omega} \to \mathbb{R}$ be the spatial concentration of firms. Assume that $\nu$ is differentiable and bounded above and below by two positive constants.

Then, the labour demand at a certain location $y \in \overline{\Omega}$, where the wage is $v > 0$, is given by

$$L(y, v) = \nu(y) l^*(v) \tag{19}$$

and satisfies the conditions of Assumption 3.1.

Assumption 3.2. *The transport cost function $c$ belongs to $C^1(\Omega^2, \mathbb{R}_+)$ and we have $c(z, z) = 0$ for all $z \in \overline{\Omega}$.*

The following theorem is the main result of the paper.

Theorem 3.1. *Under Assumptions 3.1 and 3.2, there exists at least one equilibrium, in the sense of Definition 2.2.*

The proof, presented in the following subsection, is inspired by [Achdou et al., 2022]. It relies on a fixed-point argument: we build a continuous map $\mathcal{Y}(\cdot)$ such that the fixed-points of $\mathcal{Y}$ are exactly the solutions of the equilibrium problem. To that end, we first show that the distribution of residents, $\mu$, can be explicitly obtained from the wage function $w(\cdot)$ and the pollution $\tilde{E}(\cdot)$. We then show that the solutions $w(\cdot)$ and $\tilde{E}(\cdot)$ belong to convex and compact subsets of, respectively, $C(\overline{\Omega}, \mathbb{R})$ and $L^2(\Omega)$. We then apply Schauder fixed point theorem:

Theorem (Schauder fixed point theorem). *Let $F$ be a normed vector space, $K$ a convex and compact subset of $F$ and $\mathcal{Y}$ a continuous application from $K$ into itself. Then $\mathcal{Y}$ admits at least one fixed-point.*

[4]Namely, $f(0) = 0$, $f$ is concave on $_+$, $\lim_{x \to \infty} f'(x) = 0$ and $\lim_{x \to 0} f'(x) = \infty$.

## 3.2 Fixed point

As usual in quantitative urban models [Diamond, 2016], we first make use of the housing market clearing condition and free mobility of the agents to obtain an explicit formulation of the equilibrium distribution of residents.

Lemma 3.1. *Let* $(w(\cdot), Q(\cdot), E(\cdot), \mu) \in C(\overline{\Omega}, \mathbb{R}_+^*)^2 \times H_0^1(\Omega) \cap C(\overline{\Omega}, \mathbb{R}_+^*) \times \mathcal{P}_c(\overline{\Omega})$ *be an equilibrium. Then,*

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_{\overline{\Omega}} R_\sigma(y, w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}} \, dy}, \ \forall x \in \overline{\Omega} \tag{20}$$

Equation (20) displays an equilibrium relationship between the distributions of residents, wages and pollution: people tend to locate where revenues are high and pollution is low. As a consequence, if $(w(\cdot), Q(\cdot), E(\cdot), \mu)$ is an equilibrium, then the source term of the pollution equation, $f_{\mu,w}$, can be expressed as a function depending on $w$ and $E$, i.e. for all $z \in \overline{\Omega}$

$$f_{\mu,w}(z) = f_{w,E}(z) := \frac{\int_{\overline{\Omega}^2} \delta^{-1} \mathbf{1}_{z \in \Sigma(x,y)} R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}} G_\sigma(x, y, w) \, dx \, dy}{\int_{\overline{\Omega}} R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}} \, dx} \tag{21}$$

Given this new formulation of the pollution source term, we then show that any equilibrium distribution of pollution belongs to a convex and compact subset of $L^2(\Omega)$.

Proposition 3.1. *Let* $(w, E) \in C(\overline{\Omega}, \mathbb{R}_+^*) \times H_0^1(\Omega) \cap C(\overline{\Omega}, \mathbb{R}_+^*)$. *The PDE*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_{w,E}(z), \ \forall z \in \Omega \\ u(s) = 0, \ \forall s \in \partial\Omega \end{cases} \tag{22}$$

*admits a unique solution* $u_{w,E} \in H_0^1(\Omega)$. *Moreover,* $u_{w,E}$ *is positive and belongs to*

$$K_2 = \left\{ u \in H_0^1(\Omega), \ \|\nabla u\|_{L^2} \leq |\Omega| \, \delta^{-2} \min(1, \lambda)^{-1} \right\}$$

*which is convex and compact in* $L^2(\Omega)$.

The proof relies on applying Riesz's representation theorem for the existence and uniqueness part, Hölder inequality for the majoration of the solution derivative, and Rellich's theorem for the compactness of $K_2$.

Finally, we show that any equilibrium wage map belongs to a convex and compact subset of $C(\overline{\Omega}, \mathbb{R})$. To that end, following [Petit, 2022], we show that any solution to (12) can be expressed as the unique solution to a convex minimization problem.

Proposition 3.2. *For any distribution of agents* $\mu \in \mathcal{P}_c(\overline{\Omega})$, *(12) holds for* $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ *if and only if* $w$ *is the unique minimizer of*

$$\min_{z \in C(\overline{\Omega}, \mathbb{R}_+^*)} \left\{ \phi_\mu(z) - \int_{y \in \Omega} \int_{s=\varepsilon}^{z(y)} L(y, s) ds dy \right\} \tag{23}$$

*where* $\varepsilon > 0$ *is given and*

$$\phi_\mu(z) = \int_{\overline{\Omega}} R_\sigma(x, z) d\mu(x)$$

*Moreover, the minimizer belongs to a subset* $K_1$ *convex and compact in* $C(\overline{\Omega}, \mathbb{R})$ *and independent of* $\mu$.

The outline of the proof is as follows. We first provide a priori bounds on the solutions to (23) and their derivatives to reduce the minimization problem to a compact and convex subset of $C(\overline{\Omega}, \mathbb{R})$. We then apply the direct method in the calculus of variations to deduce the existence of a unique solution.

*Remark.* The second term in minimization problem (23) can be interpreted as an integral of profits. To observe this, let us take the notations of Example 1: $L(y, v) := \nu(y) f'^{-1}(v)$, where $\nu$ is the concentration of firms and $f$ a production function. We obtain, by a change of variable in the integral

$$\int_\Omega \int_{s=\varepsilon}^{z(y)} L(y,s) ds dy = \int_\Omega \nu(y) \left[\ell(w(y))w(y) - f(\ell(w(y)))\right] dy$$

where, for any wage $v > 0$, $\ell(v) := f'^{-1}(v)$ is the labour demand of an individual firm. Thus, problem (23) is equivalent to

$$\min_{z \in \mathring{K}_0} \left\{ \phi_\mu(z) + \int_{y \in \Omega} \nu(y) \pi(z(y))\, dy \right\} \tag{24}$$

where, for any wage $v > 0$, $\pi(v) := f(\ell(v)) - \ell(v)v$ is the profit of an individual firm.

We shall now use a fixed-point argument to establish the existence of an equilibrium. Proposition 3.3 builds a map $\mathcal{Y}$ defined on $K_1 \times K_2$ whose fixed-points are exactly the equilibria.

Proposition 3.3. *Let us define the function $\mathcal{Y} : K_1 \times K_2 \longrightarrow K_1 \times K_2$ by the following construction:*

*(1) To any $(w, E) \in K_1 \times K_2$, we associate the probability $\mu(w, E)$ on $\Omega$ with density*

$$\Omega \ni x \mapsto \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\, dy} \tag{25}$$

*with respect to the Lebesgue measure,*

*(2) We define $\mathcal{Y}_1(w, E)$ as the unique solution to (23) associated to $\mu(w, E)$, i.e. $\mathcal{Y}_1(w, E)$ is the unique minimizer of*

$$\min_{z \in \mathring{K}_0} \left\{ \phi_{\mu(w,E)}(z) - \int_{y \in \overline{\Omega}} \int_{s=\varepsilon}^{z(y)} L(y,s) ds dy \right\}$$

*(3) We define $\mathcal{Y}_2(w, E)$ as the unique solution to (22), i.e.*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_{w,E}(z), \ \forall z \in \Omega \\ u(s) = 0, \ \forall s \in \partial\Omega \end{cases}$$

*The fixed points of $\mathcal{Y}$ are exactly the equilibria, in the sense of Definition 2.2.*

Proof. First, $\mathcal{Y}$ is well defined because the solutions to (23) and (22) respectively belong to $K_1$ and $K_2$. Now, if $(w, E) = \mathcal{Y}(w, E)$, let us consider $\mu$ the probability measure given by (25), and the rental price $Q : \overline{\Omega} \longrightarrow \mathbb{R}_+^*$ given by

$$Q(x) = (1-\theta) R_\sigma(x,w) \mu(x), \ \forall x \in \Omega$$

The quadruplet $(w(\cdot), Q(\cdot), E(\cdot), \mu) \in K_1 \times C(\overline{\Omega}, \mathbb{R}_+^*) \times K_2 \times (\mathcal{P}_c(\overline{\Omega}) \cap C(\overline{\Omega}))$ is an equilibrium since (12) holds because of $w = \mathcal{Y}_1(w, E)$ and Lemma 3.2, (13) holds by definition of $Q$, (14) holds because $E = \mathcal{Y}_2(w, E)$, and for all $x \in \Omega$,

$$U_{\theta,\gamma}(R_\sigma(x,w), Q(x), E(x)) = \theta^\theta \left( \int_\Omega R_\sigma(z,w)^{\frac{\theta}{1-\theta}} \tilde{E}(z)^{\frac{-\gamma}{1-\theta}} dz \right)^{1-\theta}$$

a constant value which implies that the mobility condition (15) holds. Finally, if $(w(\cdot), Q(\cdot), E(\cdot), \mu)$ is an equilibrium, from Proposition 3.1 $\mu$ is given by (25), $w$ is the solution to (23) associated to $\mu$, and $E$ is the unique solution to (22) associated to $(w, E)$. Therefore $(w, E) = \mathcal{Y}(w, E)$. □

The mapping $\mathcal{Y}$ takes only two arguments: $w$ and $E$. These two variables are enough to characterize an equilibrium, because equations (7) and (20) relate them with $Q$ and $\mu$. Proposition 3.4 establishes continuity of $\mathcal{Y}$.

Proposition 3.4. *The map $\mathcal{Y}$ is continuous on $(K_1, \|\cdot\|_{L^\infty}) \times (K_2, \|\cdot\|_{L^2})$.*

The outline of the proof is as follows. To establish the continuity of $\mathcal{Y}_1$, we first prove the continuity of the equilibrium distribution of residents, explicitly given by (3.1), with respect to $w$ and $E$. Then, we prove the continuity of the solutions to problem (23) with respect to $\mu$. To establish the continuity of $\mathcal{Y}_2$, we first prove the continuity of the source term $f_{w,E}$ with respect to $w$ and $E$, and then prove the continuity of the solutions to the scalar transport equation (22) with respect to the source term.

We are now able to prove our main theorem. We recall it here:

Theorem. *There exists at least one equilibrium, in the sense of Definition 2.2.*

Proof. By Proposition 3.4, the map $\mathcal{Y}$ is continuous from the convex and compact set $K_1 \times K_2$ into itself. By Schauder's fixed-point theorem, $\mathcal{Y}$ admits at least one fixed-point. Therefore, by Proposition 3.3, there exists at least one equilibrium, in the sense of Definition 2.2. □

## 4 NUMERICAL ASPECTS

### 4.1 Algorithm

In Section 3, we characterized an equilibrium as a fixed point of a specific map. The existence of a fixed point relied on Schauder theorem: we showed that this map is continuous, and that the equilibrium is in a convex and compact subset. But we are not certain that the map is non-expansive; probably, it is not. In the most rigorous approach, we would therefore not be allowed to apply a Banach fixed-point iterative method to compute the solution. Unfortunately, few other algorithms exist, and even fewer are adapted to Schauder theorem assumptions. They are often complex and not easily tractable from one application to another.

This is why we decided, anyway, to use a Banach fixed-point iterative method, although our problem does not satisfy the required assumptions. As a consequence, the convergence of our algorithm is not theoretically ensured. Besides, since we have not proved the uniqueness of the solution, several equilibria may exist, while not being captured by the algorithm. Yet, the many numerical experiments we have performed, with different initial conditions, seem to indicate that it always converges to a unique equilibrium.

Our algorithm is as follows. First, we initialize the pollution and wage distributions. Then, at each step, given a current wage function $w$ and pollution distribution $E$, the algorithm successively:

(1) Computes a new wage function $w^*$ which clears the labour market (12), where $E$ is given and $\mu$ obtained from formula (20);

(2) Computes a new pollution distribution $E^*$ which solves the dispersion equation (14), where $w$ is given and $\mu$ obtained from formula (20);

(3) Computes the residual $r = \|w^* - w\|_{L^\infty} + \|E^* - E\|_{L^2}$.

The algorithm iterates as long as $r$ is greater than a certain (small) arbitrary parameter $\varepsilon$. At the output of the loop, we get equilibrium values for $w$ and $E$. The population distribution $\mu$ and rental price function $Q$ are then respectively recovered from equations (20) and (13).

**ALGORITHM 1:** Equilibrium computation

**Initialize:** $E \leftarrow E^{(0)}$, $w \leftarrow w^{(0)}$, $r \leftarrow r^{(0)}$

**while** $r > \varepsilon$

1. Wage update: compute $w^*$ as the unique solution to

$$\int_\Omega G_\sigma(x, y, w^*)\mu_{w^*,E}(x)\,dx = L(y, w^*(y)),\ \forall y \in \overline{\Omega}$$

where $\mu_{w^*,E}$ is given by (20)

2. Pollution update: compute $E^*$ as the unique solution to

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_{w,E}(z),\ \forall z \in \Omega \\ u(s) = 0,\ \forall s \in \partial\Omega \end{cases} \tag{26}$$

3. Residual and new values update: $r \leftarrow \|w^* - w\|_{L^\infty} + \|E^* - E\|_{L^2}$; $w \leftarrow w^*$; $E \leftarrow E^*$

**end while**

Compute $\mu$ with

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y, w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\,dy},\ \forall x \in \overline{\Omega} \tag{27}$$

Compute $Q$ with

$$Q(x) = (1 - \theta)R_\sigma(x, w)\mu(x)\ \forall x \in \overline{\Omega},$$

**Output:** $w, Q, E, \mu$.

### 4.2 Methods

*Spatial discretization.* We consider a regular grid of our domain $\overline{\Omega}$. For writing convenience, we focus on the case where $\overline{\Omega} = [0, 1]$. Let $X_h$ be a uniform grid on $\overline{\Omega}$ with step $h := 1/N_h$, $N_h \in \mathbb{N}^*$. The points of the grids are denoted by $x_j := jh$, for $j = 0, ..., N_h$. The same grid is used to approximate both labour equation (12) and pollution dispersion equation (14). It would make sense to use a finer one for the dispersion equation, but this would involve further numerical complications that we prefer leaving for future research. The wage, rental price, pollution and residents distribution take the form of $N_h$-uplets $(w_i)$, $(Q_i)$, $(E_i)$ and $(\mu_i)$, belonging to $(0; +\infty)^{N_h}$. All the integrals are approximated with the rectangle rule.

*Economic equilibrium.* Labour equation (12) is discretized as follows

$$\sum_{i=0}^{N_h-1} h\,G_\sigma(x_i, x_j, w)\,\mu_i = L(x_j, w_j),\ \forall j \in \{0, ..., N_h\} \tag{28}$$

where

$$G_\sigma(x_i, x_j, w) = \frac{e^{\frac{w_j - c(x_i,x_j)}{\sigma}}}{\sum_{k=0}^{N_h-1} h\,e^{\frac{w_k - c(x_i,x_k)}{\sigma}}},\ \mu_i = \frac{R_\sigma(x_i, w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_i^{-\frac{\gamma}{1-\theta}}}{\sum_{k=0}^{N_h-1} R_\sigma(x_k, w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_k^{-\frac{\gamma}{1-\theta}}},\ R_\sigma(x_i, w) = \sigma \ln\left(\sum_{k=0}^{N_h-1} e^{\frac{w_k - c(x_i,x_k)}{\sigma}}\right)$$

To solve the nonlinear system (28), we use the method *scipy.optimize.root* contained in the library "Scipy" of Python, which is based on the Powell hybrid method ([Powell, 1970]).

*Pollution dispersion.* To obtain a solution for (14), our strategy is to numerically simulate the stationary solution to

$$\begin{cases} \partial_t u(z,t) - \Delta u(z,t) + \mathbf{V}(z)\cdot\nabla u(z,t) + \lambda u(z,t) = f_{w,E}(z), \ \forall (z,t) \in \Omega\times\mathbb{R}_+ \\ u(s,t) = 0, \ \forall (s,t) \in \partial\Omega\times\mathbb{R}_+ \\ u(z,0) = E(z), \ \forall z \in \Omega \end{cases} \tag{29}$$

Let $\tau > 0$ be the time step. The solution $u$ is discretized in time and space, such that $u_{j,n} = u(jh, n\tau)$, for all $(j,n) \in \{0,...,N_h\}\times\mathbb{N}$. To discretize equation (29), we use an explicit finite-difference scheme. We approximate the derivatives as follows

$$(\partial_t u)_{j,n} \approx \frac{u_{j,n+1} - u_{j,n}}{\tau} \quad (\nabla u)_{j,n} \approx \frac{u_{j+1,n} - u_{j-1,n}}{2h} \quad (\Delta u)_{j,n} \approx \frac{u_{j-1,n} - 2u_{j,n} + u_{j+1,n}}{h^2}$$

Thus, the scheme takes the following form

$$\frac{u_{j,n+1} - u_{j,n}}{\tau} = \frac{u_{j-1,n} - 2u_{j,n} + u_{j+1,n}}{h^2} - V_j\,\frac{u_{j+1,n} - u_{j-1,n}}{2h} - \lambda u_{j,n} + f_{w,E,j}$$

with the following initial and boundary conditions:

- For all $j \in \{0,...,N_h\}$, $u_{j,0} = E_j$
- For all $n \in \mathbb{N}$, $u_{0,n} = 0$ and $u_{N_h,n} = 0$ (if homogenous Dirichlet boundary conditions), or $u_{0,n} = u_{1,n}$ and $u_{N_h,n} = u_{N_h-1,n}$ (if homogenous Neumann boundary conditions).[5]

This scheme is first-order accurate in time, and second-order in space. To ensure stability, the Courant-Friedrichs–Lewy condition must hold: $V\tau/h \leq 1$.

### 4.3 Calibration

In the following simulations, we will assume that that the demand for labour has the form given in Example 1, with the production function $f(l) = l^\beta$, where $\beta \in [0,1]$ and $l \geq 0$ is the labour factor. The density of firms $\nu(y)$ can take on many forms. In the context of a monocentric city, there is typically one Central Business District (CBD), and the density of firms decreases as the distance from the CBD increases: in the first case, that we call "classic monocentric city", the CBD is located in the geographical center of the city; in the second case, called "shifted monocentric city", it is located in the west of the city (Figure 1).

Finally, we assume that the transportation cost is linear: $c(x,y) = c_0\|x-y\|, \ \forall x,y \in \overline{\Omega}$.

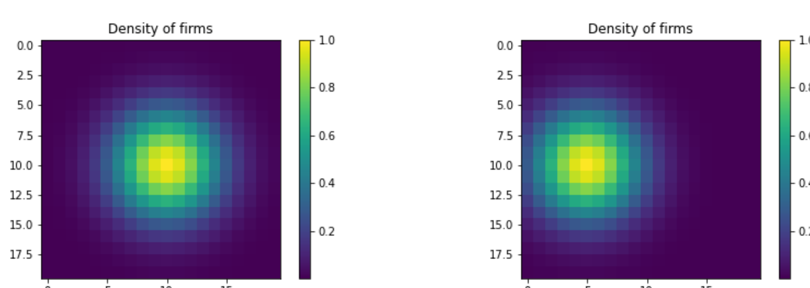



Fig. 1. Spatial concentration of firms in the 2D "classic monocentric city" (left), and the 2D "shifted monocentric city" (right).

All our baseline parameters are given in Table 1.

[5]In the following simulations, we will indeed use Neumann (instead of Dirichlet) boundary conditions, in order to obtain more realistic numerical results.

| Parameter | Symbol | Value |
|---|---|---|
| Domain | $\overline{\Omega}$ | $[0;20] \times [0;20]$ |
| Consumption-housing substitution | $\theta$ | 0.75 |
| Aversion to pollution | $\gamma$ | 0.5 |
| Transportation costs | $c_0$ | 1.0 |
| Noise on effective salaries | $\sigma$ | 0.15 |
| Capital-labour substitution | $\beta$ | 0.7 |
| Number of firms | $\nu_0$ | 1.0 |
| Background pollution level | $E_0$ | 0.1 |
| Wind | $\mathbf{V}$ | [0.0,0.0] |
| Diffusion | $k$ | 1.0 |
| Lessivage | $\lambda$ | 1.0 |
| Discretization space step | $h_x$ and $h_y$ | $h_x = 1.0$, $h_y = 1.0$ |
| Discretization time step | $\tau$ | 0.1 |
| Residual relative threshold | $\varepsilon$ | 0.05 |

Table 1. Baseline parameters.

## 5 APPLICATIONS

In this section, we aim to perform analytical and numerical simulations of our model to emphasize the effect of pollution aversion and wind on equilibrium. We will also discuss considerations related to welfare.

### 5.1 The role of pollution aversion

In order to assess the role of pollution aversion, we first apply our model to a simple case: the linear monocentric city. We also neglect the effects of diffusion and wind. These approximations allow us to obtain, in this one dimensional case, an analytical expression of the equilibrium that highlights the role played by this parameter.

Let us consider the segment $[0; 1]$ as our linear city. There is only one working place, located in 1.[6] The wage function then reduces to one single value, $w^*$, which is solution to a simple labour equation

$$L(w^*) = 1 \tag{30}$$

The revenue of an agent located in $x \in [0; 1]$ is $R(x, w^*) = w^* - c(x)$, where the function $c$ is smooth, decreasing and $c(1) = 0$. The housing equation then writes, for all $x \in [0; 1]$

$$(1 - \theta)R(x, w^*)\mu(x) = Q(x) \tag{31}$$

As for pollution, we assume, to obtain analytical results, that there is neither diffusion nor advection effect. Thus,

$$\lambda \tilde{E}(x) = \lambda E_0 + f_\mu(x)$$

for all $x \in [0; 1]$, with $E_0 > 0$. The source term $f_\mu(x)$ is the density of people commuting by $x$. As all the agents work in 1, we have

$$f_\mu(x) = \int_0^x \mu(s)\, ds$$

[6]This case does not fit exactly into the previous analytical framework where we assumed that the firms were continuously distributed over the whole domain. But the results, especially the explicit formulation of the distribution of residents, are still valid. We refer to [Achdou et al., 2022] and [Petit, 2022] where the number of workplaces is finite.

and the pollution equation becomes

$$\tilde{E}(x) = E_0 + \lambda^{-1} \int_0^x \mu(s)\, ds \tag{32}$$

Finally, the mobility condition writes

$$\operatorname{supp} \mu \subset \underset{x \in [0,1]}{\operatorname{argmax}}\ U_{\theta,\gamma}(R_\sigma(x,w), Q(x), E(x)) \tag{33}$$

Any solution $(w^*, Q(\cdot), E(\cdot), \mu) \in \mathbb{R}_+^* \times C([0;1], \mathbb{R}_+^*) \times C^1([0;1], \mathbb{R}_+^*) \times \mathcal{P}_c([0;1])$ to the system given by (30), (31), (32) and (33) is an equilibrium of our problem.

The following Proposition gives existence and uniqueness of the equilibrium. In addition, it says that pollution increases with agents' aversion for pollution.

Proposition 5.1. *The system formed by (30), (31), (32) and (33) admits a unique equilibrium, where the pollution is explicitly given by*

$$\tilde{E}(x) = \left[ E_0^{\frac{1+\gamma-\theta}{1-\theta}} + \left( \left(E_0 + \lambda^{-1}\right)^{\frac{1+\gamma-\theta}{1-\theta}} - E_0^{\frac{1+\gamma-\theta}{1-\theta}} \right) \frac{\int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds} \right]^{\frac{1-\theta}{1+\gamma-\theta}}, \ \forall x \in [0;1] \tag{34}$$

*If $\lambda$ is small enough, for all $x \in [0;1]$, $d\tilde{E}(x)/d\gamma > 0$. In other words, at the equilibrium, pollution increases with the aversion of the population to pollution, $\gamma$.*

The intuition behind Proposition 5.1 is simple. The more individuals are pollution averse, the more they tend to move away from it by living far from the city center. But, in doing so, they increase their commuting distance, thus the amount of pollution they release.

We now perform numerical simulations in the 1D and the 2D cases to illustrate this effect.

In the 1D case, we compute the equilibrium in pollution, residence, and rental price, for several values of $\gamma$, ranging from 0 to 1. We set $w^* = 1.0$. We assume a linear commuting cost function, $c(x) := c_0(1-x)$, with $c_0 = 0.3$, and set other parameters as given in Table 1: $\theta = 0.75$, $\lambda = 1.0$, $E_0 = 1.0$. Figure 2 displays the numerical results. As $\gamma$ increases, agents tends to concentrate in 0, away from the city center in 1, raising in turn the total amount of pollutants released.

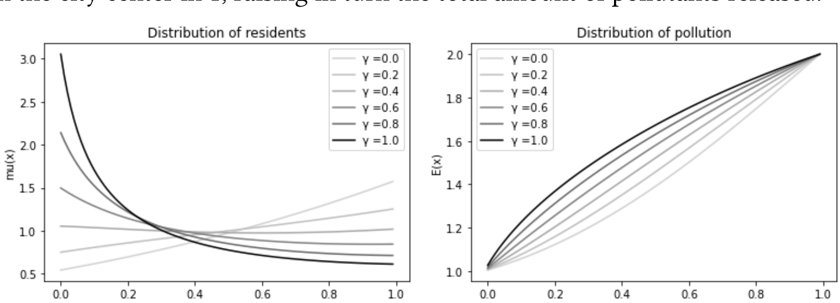



Fig. 2. Equilibrium in the 1D monocentric city, for different values of $\gamma$.

In the 2D "classic monocentric city" case (Figure 1, left), jobs do not concentrate in one point only, but the density of firms progressively declines with the distance to the city center. We also account for diffusion, and we assume homogenous Neumann boundary conditions on pollution. We compute the equilibrium given by equations (12), (13), (14) and (15). Figure 3 displays the numerical results.

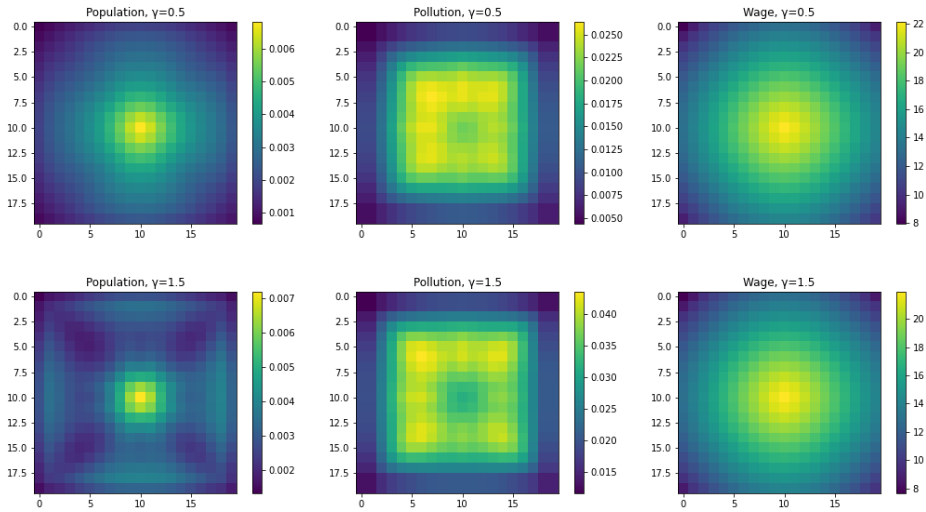



Fig. 3. Equilibrium in population, pollution and wage in the 2D classic monocentric city, for $\gamma = 0.5$ (top) and $\gamma = 1.5$ (bottom).

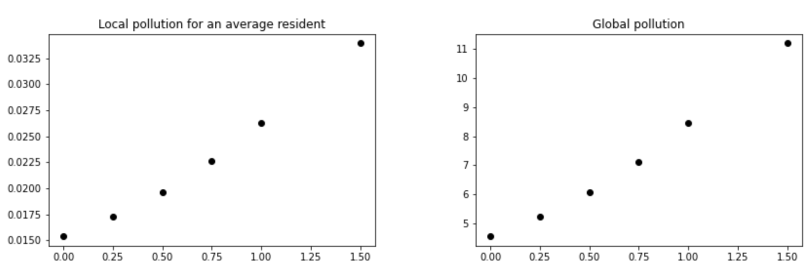



Fig. 4. Local and global pollution in the monocentric city, for different values of $\gamma$. Local pollution refers to the integral $\int_\Omega \tilde{E}\mu$, and global pollution to the integral $\int_\Omega \tilde{E}$.

We see that as $\gamma$ increases, residents become more concentrated in the periphery of the city: they tend to choose locations where pollution is small (Figure 4). As a consequence, commuting pollution rises. However, contrary to the 1D case, we observe an intermediate area, between the city center and the periphery, which is more polluted and less densely populated. This is because in the 2D case, there is no longer one working place, but a continuous distribution of firms from the more productive city center to the less productive periphery. The periphery is only polluted by its own residents, as there are no other people who commute through this area with low wages. Similarly, the city center, densely populated, is mainly polluted by its own residents, who alone satisfy a sizeable part of the demand for labour. On the contrary, the intermediate area is both polluted by its own residents, and also by the ones living in the periphery who work closer to the city center. This explains the higher concentration of air pollution in this area.

## 5.2 The role of meteorological factors

We aim at estimating the overall amount of pollution released, i.e. the quantity

$$\int_\Omega E(x)\,dx \tag{35}$$

and the pollution suffered by an average resident, i.e. the quantity

$$\int_\Omega E(x)\mu(x)\,dx \tag{36}$$

The following Proposition shows that the overall amount of pollution released is proportional to the average commuting distance traveled by an agent.

Proposition 5.2. *At the equilibrium,*

$$\lambda \int_\Omega E(x)\,dx = \mathbb{E}\left[|X-Y|\right] \tag{37}$$

*where the couple $(X,Y)$ follows the joint distribution of density $m(x,y) := \mu(x)G_\sigma(x,y,w)$.*

*Remark.* If we replace the homogenous Dirichlet by a Neumann boundary condition, we have

$$\lambda \int_\Omega E(x)\,dx = \mathbb{E}\left[|X-Y|\right] - \int_{\partial\Omega} E(s)\mathbf{V}(x)\cdot\mathbf{n}\,ds$$

where $\mathbf{n}$ is the unit vector normal to the boundary $\partial\Omega$. The additional term represents the pollution that is conveyed out of the domain by wind.

We now turn to the pollution suffered by an average resident.

Proposition 5.3. *At the equilibrium,*

$$\begin{aligned}\lambda \int_\Omega E(x)\mu(x)\,dx &= \underbrace{\int_\Omega f_{\mu,w}(x)\,\mu(x)\,dx}_{source\ term} - \underbrace{\int_\Omega \nabla E(x)\cdot\nabla\mu(x)\,dx}_{diffusion\ term} \\ &\quad + \underbrace{\frac{\theta}{1-\theta+\gamma}\int_\Omega \left[\mathbf{V}(x)\cdot\nabla R_\sigma(x,w)\right]E(x)\,\mu(x)\,dx}_{advection\ term}\end{aligned}$$

*Remark.* If we replace the homogenous Dirichlet by an homogenous Neumann boundary condition, we have

$$\begin{aligned}\lambda \int_\Omega E(x)\mu(x)\,dx &= \int_\Omega f_{\mu,w}(x)\,\mu(x)\,dx - \int_\Omega \nabla E(x)\cdot\nabla\mu(x)\,dx \\ &\quad + \frac{\theta}{1-\theta+\gamma}\int_\Omega \left[\mathbf{V}(x)\cdot\nabla R_\sigma(x,w)\right]E(x)\,\mu(x)\,dx \\ &\quad - \frac{1-\theta}{1-\theta+\gamma}\int_{\partial\Omega} E(s)\mu(s)\mathbf{V}(s)\cdot\mathbf{n}\,ds\end{aligned}$$

where $\mathbf{n}$ is the unit vector normal to the boundary $\partial\Omega$. The additional term represents the pollution that is conveyed out of the domain by wind.

Proposition 5.3 has an interesting interpretation that combines both economic and meteorological factors. It says that the air quality experienced by an average resident can be decomposed into three terms: a *source term*, a *diffusion term* and an *advection term*.

The source term refers to the pollution emitted by the cars at the resident's location, just out front her house. It depends on the automobile traffic at this location, and therefore, on where people work and live.

The diffusion term refers to the movement of pollution from surrounding areas through diffusion processes. It occurs due to the concentration gradients present in the atmosphere, with pollution spreading from areas of high concentration to areas of low concentration. If the gradients of pollution ($\nabla E$) and residents ($\nabla \mu$) are oriented in the same direction ($\nabla E \cdot \nabla \mu \geq 0$), pollution diffuses from areas of high population density to areas with less population density, which tends to decrease the pollution experienced by residents. However, if the gradients of pollution and population density are in opposite directions ($\nabla E \cdot \nabla \mu \leq 0$), pollution diffuses from areas of low population density to areas with high population density, which tends to increase the pollution experienced by residents.

The advection term refers to the movement of pollution by wind. It has an effect on the pollution suffered by the residents, depending on the relative orientation of wind ($\mathbf{V}$) and revenue gradient ($\nabla R_\sigma(\cdot, w)$). If $\mathbf{V} \cdot \nabla R_\sigma(\cdot, w)$ is positive, the wind is carrying pollution towards areas with high revenue, where people tend to concentrate, resulting in increased pollution for residents. On the other hand, if $\mathbf{V} \cdot \nabla R_\sigma(\cdot, w)$ is negative, the wind is carrying pollution away from high population density areas, reducing the pollution experienced by residents.

We conduct numerical simulations in the 2D case to illustrate the role of wind. The city is assumed to have a monocentric structure, but with the business district shifted to the West (Figure 1, right). We consider two types of wind, constant across the city: a West-East wind with velocity of $\mathbf{V} = (4.5; 0)$ and an East-West wind with velocity of $\mathbf{V} = (-4.5; 0)$. We set $\gamma = 1.5$, and other parameters as given in Table 1. We assume homogenous Neumann boundary conditions on pollution. Figure 5 displays the resulting equilibria in pollution and population for the different wind regimes.

When the wind blows from an East-West direction (Figure 5, middle), it pushes pollution towards the business district. In this case, the direction of the wind aligns with the direction of the revenue gradient. As a result, we see that people tend to live farther away from the business district, leading to increased overall pollution levels. Moreover, in line with Proposition 5.3, it also contributes to increase the contamination suffered by an average resident (Figure 6). For the same reasons as mentioned in subsection 5.1, we also observe the presence of a middle area, between the business district and the periphery, which is more polluted and less densely populated.

When the wind blows from a West-East direction (Figure 5, right), it pushes pollution away from the business district. In this case, the direction of the wind is opposite to the direction of the revenue gradient. As a result, people tend to live closer to the business district, leading to decreased overall pollution levels. In line with Proposition 5.3, the wind also contributes to reduce the contamination suffered by an average resident (Figure 6).

We observe that the East-West wind has a greater impact on total pollution released compared to the West-East wind. This is because in the former case, the wind carries pollution to a more densely populated area, resulting in larger population movements to escape the pollution. This effect diminishes as the sensitivity of people to air pollution, $\gamma$, decreases. When $\gamma = 0$, the wind has no influence on the amount of pollution released.

### 5.3 Welfare aspects

*5.3.1 Labour market.* Is the labour market efficient in our model? For any distribution of residents $\mu \in \mathcal{P}_c(\overline{\Omega})$, the equilibrium in the labour market is given by equation (12). We assume that the labour demand function has the form given in Example 1: $L(y, v) := \nu(y) f'^{-1}(v)$, where $\nu$ is the concentration of firms and $f$ the production function of an individual firm. Proposition 5.4 shows

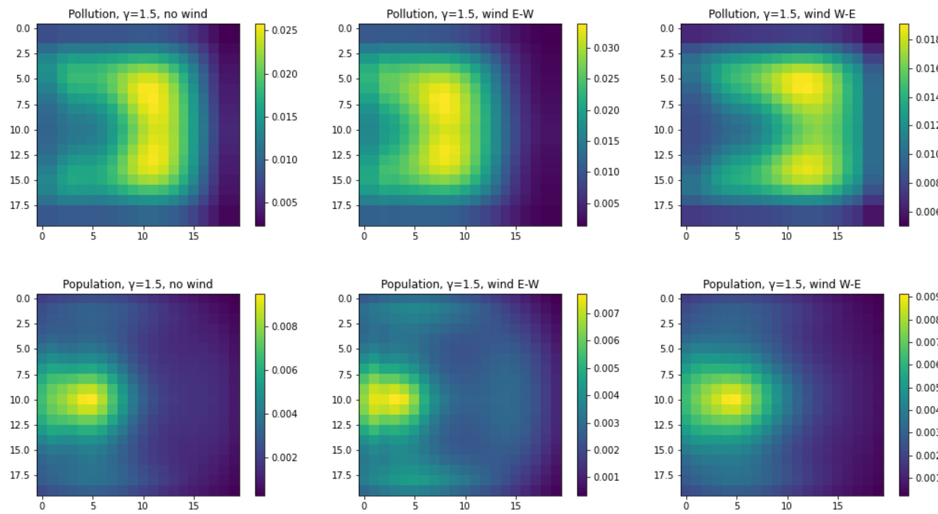



Fig. 5. Equilibrium in population and pollution in the "shifted" monocentric city, $\gamma = 1.5$, without wind (left), with an East-West wind (middle), and with a West-East wind (right).

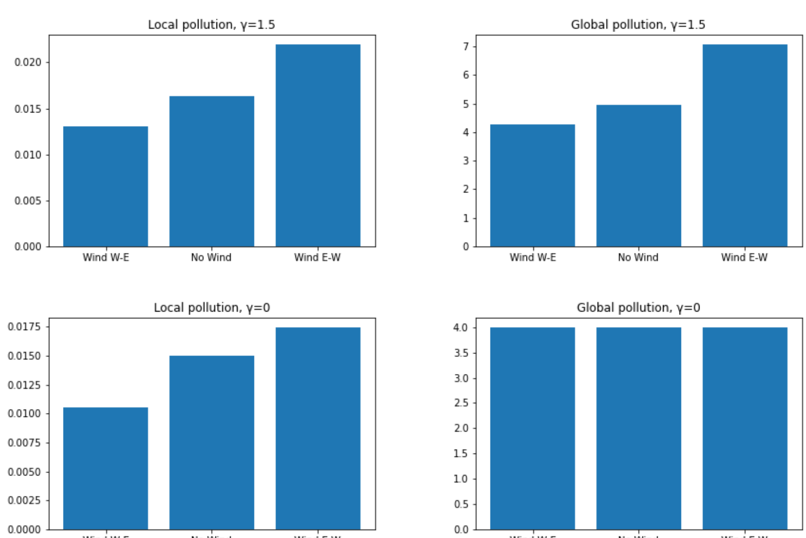



Fig. 6. Local and global pollution in the "shifted" monocentric city, for different wind regimes and values of $\gamma$. Local pollution refers to the integral $\int_\Omega \tilde{E}\mu + \frac{1-\theta}{1-\theta+\gamma} \int_{\partial\Omega} E\mu \mathbf{V} \cdot \mathbf{n}\, ds$, and global pollution to the integral $\int_\Omega f_{\mu,w}$.

that given the commuting probabilities $G_\sigma(x, y, w)$, the clearing wage maximizes the total surplus of firms and workers.

Proposition 5.4. *For any $\mu \in \mathcal{P}_c(\overline{\Omega})$, if $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ solves* (12), *then $w$ solves*

$$\sup_{w(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} \nu(y)[f(l_{\mu,w}(y)) - l_{\mu,w}(y)w(y)]dy + \int_{\overline{\Omega}} R_\sigma(x,w)d\mu(x) \right\} \tag{38}$$

*where, for all $y \in \overline{\Omega}$ and $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$*

$$l_{\mu,w}(y) := \int_{\overline{\Omega}} G_\sigma(x,y,w)d\mu(x). \tag{39}$$

This result is economically not surprising, because the labour market is not affected by the pollution externality and we assumed perfect competition between firms. Mathematically, the proof relies on applying Fenchel-Rockafellar duality theorem to the minimization problem of Proposition 3.2.

*5.3.2 Residential market.* Is the residential market efficient in our model? For any $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$, the equilibrium in the residential market is given by equations (13), (14) and (15).

At the the equilibrium, the Nash distribution of residents solves the mean-field problem

$$\sup_{m\in\mathcal{P}_c(\overline{\Omega})} \int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q(x), \tilde{E}(x))dm(x) \tag{40}$$

On the other hand, the Pareto-optimal distribution of residents should solve

$$\sup_{m\in\mathcal{P}_c(\overline{\Omega})} \int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q_m(x), \tilde{E}_m(x))dm(x) \tag{41}$$

where, given a distribution of residents $m \in \mathcal{P}_c(\overline{\Omega})$, $Q_m$ is the clearing rental price solution to (13), and $E_m$ is the pollution concentration solution to (14).

Problems (40) and (41) do not coincide in general, because contrary to the former, the latter takes into account that at the equilibrium, the rental price $Q$ and pollution concentration $E$ depend on the distribution of residents $m$.

Proposition 5.5 shows that the Nash distribution of residents also solves

$$\sup_{m\in\mathcal{P}_c(\overline{\Omega})} \int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q_m(x), \tilde{E}(x))dm(x) \tag{42}$$

Thus, without pollution externality ($\gamma = 0$), the clearing rental price leads to a Pareto equilibrium.[7] However, when $\gamma > 0$, problem (42) does not coincide with (41). In this case, the residential market is inefficient, because the agents do not internalize the effects of their location decisions on air pollution.

Proposition 5.5. *Fix any $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$. If the triplet $(Q(\cdot), E(\cdot), \mu) \in C(\overline{\Omega}, \mathbb{R}_+^*) \times \Big(H_0^1(\Omega) \cap C(\overline{\Omega}, \mathbb{R}_+^*)\Big) \times \mathcal{P}_c(\overline{\Omega})$ is solution to* (13), (14) *and* (15), *then $\mu$ solves*

$$\sup_{m\in\mathcal{P}_c(\overline{\Omega})} \int_{\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w), Q_m(x), \tilde{E}(x))m(x)dx \tag{43}$$

*where, for any distribution of residents $m \in \mathcal{P}_c(\overline{\Omega})$, $Q_m \in C(\overline{\Omega}, \mathbb{R}_+^*)$ is the clearing rental price solution to* (13).

[7]When $\gamma = 0$, the utility function does not depend on the pollution argument.

## 6 CONCLUSION

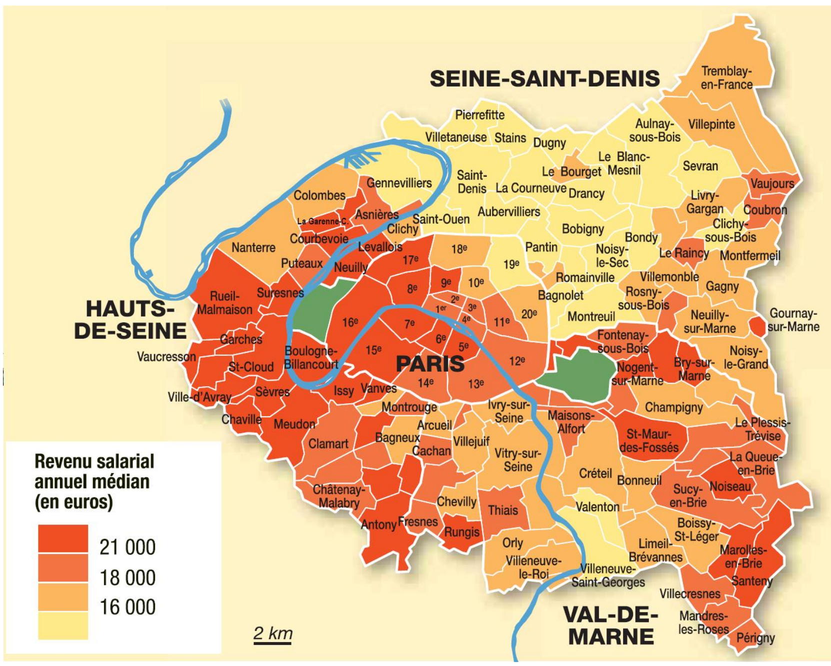
We developed an equilibrium model of city in which the labour market, the residential market and pollution are interdependent. Our model differs from existing literature in that it allows for cities of any shape and includes a realistic description of pollution dispersion.

We proved existence of equilibria and proposed an algorithm for computing solutions. We then examined various analytical and numerical applications of the model. In particular, we looked at the role of two parameters, pollution aversion and wind, on the equilibrium. We finally analyzed it from a welfare perspective.

Our results emphasize the relevance of integrating physical and economic approaches in the study of urban air pollution. They open several avenues of research, such as investigating whether the equilibrium is unique, incorporating endogenous firm location and agglomeration externalities, examining the relationship between urban pollution and inequality through agent heterogeneity, and analyzing regulatory issues such as the effects of a gasoline tax on the urban structure.

## APPENDIX 1

**Map of median annual salary revenues (in euros) in Paris and some Parisian suburbs.**



# APPENDIX 2: PROOFS

## Proof of Lemma 3.1

If $(w(\cdot), Q(\cdot), E(\cdot), \mu)$ is an equilibrium, then by equations (3) and (13) we have, for all $x \in \overline{\Omega}$

$$U_{\theta,\gamma}(R_\sigma(x,w), \mu(x), \tilde{E}(x)) = \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}}$$

Moreover, the mobility condition (15) is equivalent to: $\operatorname{supp}\mu \subset \underset{x\in\overline{\Omega}}{\operatorname{argmax}}\ U_{\theta,\gamma}(R_\sigma(x,w), \mu(x), E(x))$. This implies that there exists a real number $\beta$ such that

$$\begin{cases} U_{\theta,\gamma}(R_\sigma(x,w), \mu(x), \tilde{E}(x)) \le \beta, \ \forall x \in \overline{\Omega}, \\ U_{\theta,\gamma}(R_\sigma(x,w), \mu(x), \tilde{E}(x)) = \beta, \ \forall x \in \operatorname{supp}\mu. \end{cases}$$

By (48), for all $x \in \overline{\Omega}$, $R_\sigma(x,w) \ge R(x,w) \ge w(x) - c(x,x) \ge \ell^{-1}(\eta)$. Then

$$\theta^\theta \frac{\ell^{-1}(\eta)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}} \le \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}} \le \beta$$

This implies that for all $x \in \operatorname{supp}\mu$,

$$\theta^\theta \frac{\ell^{-1}(\eta)\ \|\tilde{E}\|_{L^\infty}^{-\gamma}}{|\beta|+1} \le \mu(x)^{1-\theta}$$

This means that $\mu$ is bounded away from zero by a positive constant. By continuity of $\mu$, we deduce that $\operatorname{supp}\mu = \overline{\Omega}$. Then

$$\theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}} = \beta, \ \forall x \in \overline{\Omega}$$

Hence

$$\mu(x) = \left(\frac{\theta^\theta}{\beta}\right)^{\frac{1}{1-\theta}} R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{\frac{-\gamma}{1-\theta}}$$

Since $\mu$ is a probability measure on $\overline{\Omega}$

$$\mu(x) = \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\, dy}, \ \forall x \in \overline{\Omega}$$

## Proof of Proposition 3.2

We first need to prove the following Lemma, which gives a regularity result about the equilibrium wage maps.

Lemma 6.1. *Fix any $\mu \in \mathcal{P}_c(\overline{\Omega})$. If $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ is a solution to (12), then $w \in C^1(\Omega, \mathbb{R}_+^*)$. Moreover, there exists $C_1, C_2 > 0$ independent of $\mu$ such that for all $y \in \Omega$*

$$|\nabla w(y)| \le e^{\|w\|_{L^\infty}}(C_1 \ell(w(y)) + C_2) \tag{44}$$

Proof. If $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ is a solution to (12), then for all $y \in \overline{\Omega}$

$$\frac{e^{\frac{w(y)}{\sigma}}}{\ell(w(y))} = \left(\frac{\int_\Omega e^{\frac{w(z)-c(x,z)}{\sigma}} dz}{\int_\Omega e^{\frac{-c(x,y)}{\sigma}} d\mu(x)}\right) \nu(y) \tag{45}$$

Let

$$h(v) := \frac{e^{\frac{v}{\sigma}}}{\ell(v)}$$

The function $h$ is a smooth bijection from $\mathbb{R}_+^*$ to $\mathbb{R}_+$. Using equation (45), we have, for all $y \in \overline{\Omega}$

$$w(y) = h^{-1}\left[\left(\frac{\int_{\overline{\Omega}} e^{\frac{w(z)-c(x,z)}{\sigma}} dz}{\int_{\overline{\Omega}} e^{\frac{-c(x,y)}{\sigma}} d\mu(x)}\right) v(y)\right]$$

which shows that $w$ is differentiable because $h^{-1}$, $c(x,\cdot)$ and $v$ are.

Now, if we differentiate equation (12), we get, for $y \in \overline{\Omega}$

$$\int_{\overline{\Omega}} \nabla_y G_\sigma(x,y,w) d\mu(x) = \nabla_y L(y, w(y)) + \partial_v L(y, w(y)) \nabla w(y)$$

i.e.

$$\int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)}{\sigma} (\nabla w(y) - \nabla_y c(x,y))\, d\mu(x) = \nabla_y L(y, w(y)) + \partial_v L(y, w(y)) \nabla w(y)$$

For all $y \in \overline{\Omega}$, $\partial_v L(y, w(y)) < 0$, then $\int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)}{\sigma}\, d\mu(x) - \partial_v L(y, w(y)) > 0$ , and therefore

$$\nabla w(y) = \frac{\nabla_y L(y, w(y)) + \int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)\nabla_y c(x,y)}{\sigma}}{\int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)}{\sigma}\, d\mu(x) - \partial_v L(y, w(y))}$$

We have

$$\left|\nabla_y L(y, w(y)) + \int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)\nabla_y c(x,y)}{\sigma}\right| \le \|\nabla v\|_{L^\infty} \ell(w(y)) + \sigma^{-1} \|\nabla_y c\|_{L^\infty}$$

and

$$\left|\int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)}{\sigma}\, d\mu(x) - \partial_v L(y, w(y))\right| \ge \int_{\overline{\Omega}} \frac{G_\sigma(x,y,w)}{\sigma}\, d\mu(x) \ge \frac{e^{-\frac{\|w\|_{L^\infty}+\|c\|_{L^\infty}}{\sigma}}}{\sigma|\Omega|}$$

Then, for all $y \in \overline{\Omega}$

$$|\nabla w(y)| \le \sigma|\Omega| e^{\frac{\|w\|_{L^\infty}+\|c\|_{L^\infty}}{\sigma}} \left(\|\nabla v\|_{L^\infty} \ell(w(y)) + \sigma^{-1} \|\nabla_y c\|_{L^\infty}\right)$$

which gives the desired estimate. □

Lemma 6.1 shows that wage maps which are solutions to (12) belong to the following subset of $C(\overline{\Omega}, \mathbb{R}_+^*)$

$$K_0 := \left\{z \in C^1(\Omega, \mathbb{R}_+^*),\ |\nabla z(y)| \le e^{\|z\|_{L^\infty}} (C_1 \ell(z(y)) + C_2)\ \forall y \in \overline{\Omega}\right\}$$

where $C_1$ and $C_2$ are given in Lemma 6.1. The subset $K_0$ is non empty (it contains the subset of constant and positive functions). Without loss of generality we can also assume that the solutions belong to the interior of $K_0$ (if not the case, expand the subset by taking an arbitrarily larger constant $C_2$).

Now, consider the map $\Lambda_\mu : \mathring{K}_0 \to \mathbb{R}$ defined by

$$\Lambda_\mu(z) = \phi_\mu(z) - \int_{y\in\overline{\Omega}} \int_{s=\varepsilon}^{z(y)} L(y,s) ds dy$$

The map $\Lambda_\mu$ is continuous on $(\mathring{K}_0, \|\cdot\|_\infty)$ and strictly convex. To show existence of a minimizer, we will provide a priori bounds on the solution and its derivative to reduce the minimization problem to a compact subset of $\mathring{K}_0$.

*First a priori bound.* Let fix $z$ and $\hat{z}$ two elements of $\mathring{K}_0$ such that

$$\Lambda_\mu(z) \leq \Lambda_\mu(\hat{z})$$

We note that

$$\Lambda_\mu(z) = \phi_\mu(z) - \int_{y\in\overline{\Omega}} \int_{s=\varepsilon}^{z(y)} L(y,s)\, ds\, dy \geq \|z\|_\infty - \|c\|_\infty - \int_{\overline{\Omega}} \int_{\varepsilon}^{z(y)} L(y,s)\, ds\, dy$$

since $R_\sigma(x,z) \geq R(x,z) \geq \|z\|_\infty - \|c\|_\infty$. Then

$$\Lambda_\mu(\hat{z}) \geq \|z\|_\infty - \|c\|_\infty - \int_{\overline{\Omega}} \int_{\varepsilon}^{z(y)} L(y,s)\, ds\, dy$$

and

$$\|z\|_\infty \leq \|c\|_\infty + \phi_\mu(\hat{z}) + \int_{\overline{\Omega}} \int_{\hat{z}(y)}^{z(y)} L(y,s)\, ds\, dy$$

Now

$$\phi_\mu(\hat{z}) \leq \|\hat{z}\|_\infty + \sigma \ln(2)$$

because $R_\sigma(x,\hat{z}) \leq R(x,\hat{z}) + \sigma\ln(2)$ and $R(x,\hat{z}) \leq \|\hat{z}\|_\infty$. Besides, due to the monotonicity and positivity of the functions $L(y,\cdot)$, we have

$$\int_{\hat{z}(y)}^{z(y)} L(y,s)\, ds \geq L(y,\hat{z}(y))(z(y)-\hat{z}(y)) \geq L(y,\hat{z}(y))z(y)$$

Finally

$$\|z\|_\infty \leq \|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2) + \|z\|_\infty \int_{\overline{\Omega}} L(y,\hat{z}(y))\, dy$$

Thus, if $z$ and $\hat{z}$ belong to $\mathring{K}_0$ and satisfy

$$\begin{cases} \Lambda_\mu(z) \leq \Lambda_\mu(\hat{z}) \\ \int_{\overline{\Omega}} L(y,\hat{z}(y))\, dy < 1 \end{cases}$$

we obtain a similar upper bound as in [Petit, 2022]

$$\|z\|_\infty \leq \frac{\|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2)}{1 - \int_{\overline{\Omega}} L(y,\hat{z}(y))\, dy}$$

As a consequence, if $z$ is a minimizer of problem (23), we have

$$\|z\|_{L^\infty} \leq M_1 \tag{46}$$

where

$$M_1 = \inf\left\{ \frac{\|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2)}{1 - \int_{\overline{\Omega}} L(y,\hat{z}(y))\, dy}, \hat{z} \in \mathring{K}_0, L(y,\hat{z}(y))\, dy < 1 \right\} \tag{47}$$

*Second a priori bound.* We are now looking for a bound from below. We claim that if $z \in \mathring{K}_0$ is a minimizer, then for all $y \in \overline{\Omega}$, $L(y,z(y)) \leq 1$, i.e. $z(y) \geq \ell^{-1}(\nu(y)^{-1})$. For $z \in \mathring{K}_0$, the Fréchet derivative of $\Lambda_\mu$ at $z$ is the following application

$$D_z\Lambda_\mu : \mathring{K}_0 \to \mathbb{R},\ h \mapsto \int_{\Omega^2} G_\sigma(x,y,z)h(y)\, dy\, d\mu(x) - \int_{\Omega} L(y,z(y))h(y)\, dy$$

Assume by contradiction that there exists $y^* \in \overline{\Omega}$, $L(y^*, z(y^*)) > 1$. As $L$ and $z$ are continuous, there exists an open ball $B_r(y^*) \subset \Omega$ with $r > 0$, such that for all $y \in B_r(y^*)$, $L(y, z(y)) > 1$. Let $h^0 \in \mathring{K}_0$ and such that $h^0_{|B_r(y^*)} > 0$ and $h^0_{|\overline{\Omega}\setminus B_r(y^*)} = 0$. Then

$$\int_{\overline{\Omega}} L(y, z(y))h^0(y)\,dy = \int_{B_r(y^*)} L(y, z(y))h^0(y)\,dy > \int_{B_r(y^*)} h^0(y)\,dy$$

and

$$\begin{aligned}\int_{\overline{\Omega}^2} G_\sigma(x, y, z)h^0(y)\,dy\,d\mu(x) = \int_{\overline{\Omega}\times B_r(y^*)} G_\sigma(x, y, z)h^0(y)\,dy\,d\mu(x) &\leq \int_{\overline{\Omega}\times B_r(y^*)} h^0(y)\,dy\,d\mu(x)\\ &\leq \int_{B_r(y^*)} h^0(y)\,dy\end{aligned}$$

Therefore

$$D_z\Lambda_\mu.h^0 < 0$$

which means that in this case, $z$ is not a minimizer. Conclusion: if $z \in \mathring{K}_0$ is a minimizer, then

$$\forall y \in \overline{\Omega},\ \ell(z(y)) \leq \nu(y)^{-1} \leq \eta \tag{48}$$

*Third a priori bound.* Now, if $z \in K_0$ satisfies the a priori bounds (46) and (48), then, by inequality (44)

$$\|\nabla z\|_{L^\infty} \leq e^{M_1}(C_1\eta + C_2)$$

meaning that we have a constant $M_2 > 0$ such that

$$\|\nabla z\|_{L^\infty} \leq M_2$$

Let us introduce the following subset of $\mathring{K}_0$

$$K_1 := \left\{z \in C^1(\Omega, \mathbb{R}_+),\ z(\cdot) \geq \ell^{-1}(\eta),\ \|z\|_{L^\infty} \leq M_1,\ \|\nabla z\|_{L^\infty} \leq M_2\right\}$$

The subset $K_1$ is convex and compact for the uniform norm $\|\cdot\|_{L^\infty}$, as a consequence of Ascoli-Arzelà theorem. We have proved that

$$\min_{z\in\mathring{K}_0} \Lambda_\mu(z) = \min_{z\in K_1} \Lambda_\mu(z)$$

*Conclusion.* Let us take a minimizing sequence $(w_n)_{n\in\mathbb{N}}$ of the problem

$$\min_{w\in K_1} \Lambda_\mu(w)$$

The compactness of $K_1$ and continuity of $\Lambda_\mu$ ensure the existence of a minimizer $w \in K_1$. The uniqueness is ensured by the strict convexity of $\Lambda_\mu$. This provides the existence and uniqueness of a solution to (23).

*Characterization of the minimizer.* Since $\Lambda_\mu$ is strictly convex and smooth, $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$ is a minimizer if and only if $D_w\Lambda_\mu = 0$, i.e. for all $y \in \overline{\Omega}$

$$\int_{\overline{\Omega}} G_\sigma(x, y, w)d\mu(x) = L(y, w(y))$$

### Proof of Proposition 3.1

For the existence and uniqueness part, we apply Riesz's representation theorem. Let us consider the following inner product, on $H_0^1(\Omega)$

$$(u,v) := \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\, v + \lambda \int_\Omega uv$$

The positive definite property of this inner product is ensured by the fact that $\lambda$ is positive, and

$$\begin{aligned}\int_\Omega (\mathbf{V}\cdot\nabla u)\, u &= \int_\Omega \mathbf{V}\cdot\nabla\left(\frac{1}{2}u\cdot u\right)\\ &= -\int_\Omega (\nabla\cdot\mathbf{V})\left(\frac{1}{2}u\cdot u\right) + \int_{\partial\Omega}\left(\frac{1}{2}u\cdot u\right)\mathbf{V}\cdot\mathbf{n}\; ds\\ &= 0\end{aligned}$$

where we first used the divergence theorem, and then the fact that $\nabla\cdot\mathbf{V}(z) = 0$ for all $z\in\Omega$, and $u(s) = 0$ for all $s\in\partial\Omega$, by equation (9). Now, consider the linear functional

$$\Lambda : H_0^1(\Omega) \longrightarrow \mathbb{R},\ v \mapsto \int_\Omega f_{w,q} v$$

Hölder inequality gives, for all $v\in H_0^1(\Omega)$

$$|\Lambda(v)| \le \|f_{w,q}\|_{L^2}\,\|v\|_{L^2}$$

Thus, $\Lambda$ is a bounded linear operator on $H_0^1(\Omega)$, thus a linear form on this Hilbert space. By Riesz's representation theorem, there exists a unique $u_{w,q}\in H_0^1(\Omega)$ such that for all $v\in H_0^1(\Omega)$, $\Lambda(v) = (u_{w,q}|v)$ i.e.

$$\int_\Omega \nabla u_{w,q}\cdot\nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\, v + \lambda\int_\Omega u_{w,q} v = \int_\Omega f_{w,q} v \tag{49}$$

The positivity of $u_{w,q}$ is a direct consequence of the maximum principle. Regarding the majoration of $\nabla u_{w,q}$ in $L^2(\Omega)$, equation (49) applied to $v = u_{w,q}$ yields

$$\|\nabla u_{w,q}\|_{L^2}^2 + \lambda\|u_{w,q}\|_{L^2}^2 \le \|f_{w,q}\|_{L^2}\,\|u_{w,q}\|_{L^2}$$

by Hölder inequality. Then

$$\min(1,\lambda)\|u_{w,q}\|_{H_0^1}^2 \le \|f_{w,q}\|_{L^2}\,\|u_{w,q}\|_{H_0^1} \le |\Omega|\,\|f_{w,q}\|_{L^\infty}^2\,\|u_{w,q}\|_{H_0^1}$$

where $\|v\|_{H_0^1} := \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)^{1/2}$ for all $v\in H_0^1(\Omega)$. Therefore

$$\|u_{w,q}\|_{H_0^1} \le |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$$

which yields

$$\|\nabla u_{w,q}\|_{L^2} \le |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$$

Convexity of $K_2$ is immediate. The proof of compactness is inspired by [Le Dret, 2013]. Let us denote $k_0 := |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$. By Rellich's theorem, the embedding $H_0^1(\Omega)\subset L^2(\Omega)$ is compact. Therefore $K_2$, which is bounded in $H_0^1(\Omega)$, is relatively compact in $L^2(\Omega)$. Let us show that $K_2$ is closed in $L^2(\Omega)$. If $(v_n)\in K_2^{\mathbb{N}}$ converges to $v\in L^2(\Omega)$, then $(v_n)$ is bounded in $H_0^1(\Omega)$ and contains a subsequence $(v_{n'})$ that converges weakly to $v'\in H_0^1(\Omega)$. By uniqueness of the limit, $v' = v$, and the lower semicontinuity of the norm implies $\|v\|_{H^1} \le \liminf_{n'\infty}\|v_{n'}\|_{H^1} \le k_0$ (here we consider the semi-norm $\|v\|_{H_0^1} := \|\nabla v\|_{L^2}$ by Poincaré inequality). Consequently, $v\in K_2$, and $K_2$ is compact.

## Proof of Proposition 3.4

*Continuity of the map $\mathcal{Y}_1$ on $(K_1, \|\cdot\|_{L^\infty})$.* We first need to prove the following two lemmas. The first one proves continuity of the equilibrium distribution of residents, explicitly given by (20), with respect to $w$ and $E$. The second one proves (weak) continuity of the solutions to problem (23) with respect to $\mu$.

Lemma 6.2.

(1) *Let $E \in K_2$, and $(w_n)$ be a sequence in $K_1$. If $\|w_n - w\|_{L^\infty} \to 0$ for some $w \in K_1$, then $\|\mu(w_n, E) - \mu(w, E)\|_{L^1} \to 0$.*
(2) *Let $w \in K_1$, and $(E_n)$ be a sequence in $K_2$. If $\|E_n - E\|_{L^2} \to 0$ for some $E \in K_2$, then $\|\mu(w, E_n) - \mu(w, E)\|_{L^1} \to 0$.*

Proof. (1) Let us denote $\mu_n := \mu(w_n, E)$ and $\mu := \mu(w, E)$. Let $x \in \Omega$. We have

$$\begin{aligned}
|\mu_n(x) - \mu(x)| &= \left| \frac{R_\sigma(x, w_n)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{\frac{-\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y, w_n)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy} - \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{\frac{-\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y, w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy} \right| \\
&= \left( \int_\Omega R_\sigma(y, w_n)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy \right)^{-1} \left( \int_\Omega R_\sigma(y, w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy \right)^{-1} \\
&\cdot \left| \int_\Omega (\tilde{E}(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} \left[ (R_\sigma(x, w_n) R_\sigma(y, w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y, w_n) R_\sigma(x, w))^{\frac{\theta}{1-\theta}} \right] dy \right| \\
&\leq (2E_0)^{\frac{-\gamma}{1-\theta}} \int_\Omega \left| (R_\sigma(x, w_n) R_\sigma(y, w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y, w_n) R_\sigma(x, w))^{\frac{\theta}{1-\theta}} \right| dy
\end{aligned}$$

Now, $R_\sigma(\cdot, w)$ is bounded from below by $R_- := \ell^{-1}(\eta)$, and from above by $R_+ := M_1 + \sigma \ln(2)$, with $M_1$ given by (47). The function $\mathbb{R} \ni a \mapsto a^{\frac{\theta}{1-\theta}}$ has continuous and bounded derivative on $[R_-^2; R_+^2]$. It is therefore Lipschitz on this segment, hence for all $y \in \Omega$,

$$\left| (R_\sigma(x, w_n) R_\sigma(y, w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y, w_n) R_\sigma(x, w))^{\frac{\theta}{1-\theta}} \right| \leq C \left| R_\sigma(x, w_n) R_\sigma(y, w) - R_\sigma(y, w_n) R_\sigma(x, w) \right|$$

for some constant $C \geq 0$. Besides

$$\begin{aligned}
|R_\sigma(x, w_n) R_\sigma(y, w) - R_\sigma(y, w_n) R_\sigma(x, w)| &\leq (R_\sigma(y, w_n) + R_\sigma(y, w)) |R_\sigma(x, w_n) - R_\sigma(x, w)| \\
&\leq 2\, R_+ \, |R_\sigma(x, w_n) - R_\sigma(x, w)| \\
&\leq C \|w - w_n\|_{L^\infty}
\end{aligned}$$

for another $C \geq 0$. The majoration comes from the fact that $\ln(\cdot)$ and $\exp(\cdot)$ are Lipschitz on compact subsets of, respectively, $\mathbb{R}_+^*$ and $\mathbb{R}$. Thus

$$|\mu_n(x) - \mu(x)| \leq C \|w - w_n\|_{L^\infty}$$

for another $C \geq 0$. This gives the $L^1$ convergence of $(\mu_n)$ to $\mu$.

(2) Let us denote $\mu_n := \mu(w, E_n)$ and $\mu := \mu(w, E)$. For all $x \in \Omega$, we have

$$
\begin{aligned}
|\mu_n(x) - \mu(x)| &= \left| \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E_n}(x)^{\frac{-\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E_n}(y)^{\frac{-\gamma}{1-\theta}}\, dy} - \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{\frac{-\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy} \right| \\
&= \left( \int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E_n}(y)^{\frac{-\gamma}{1-\theta}}\, dy \right)^{-1} \left( \int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\, dy \right)^{-1} \\
&\cdot \left| \int_\Omega (R_\sigma(x,w) R_\sigma(y,w))^{\frac{\theta}{1-\theta}} \left[ (\tilde{E_n}(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E_n}(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right] dy \right| \\
&\leq C \int_\Omega \left| (\tilde{E_n}(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E_n}(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right| dy
\end{aligned}
$$

for some constant $C \geq 0$. The majoration comes from the fact $\|R_\sigma(\cdot, w)\|_{L^\infty} \leq M_1 + \sigma \ln(2)$, with $M_1$ given by (47), and that $\tilde{E_n}$ and $\tilde{E}$ are minored by $E_0 > 0$. Now, the function $\mathbb{R} \ni a \mapsto a^{\frac{-\gamma}{1-\theta}}$ has continuous and bounded derivative on $[E_0^2; +\infty)$ and is therefore Lipschitz on this interval, hence for all $x, y \in \Omega$,

$$
\left| (\tilde{E_n}(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E_n}(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right| \leq C |\tilde{E_n}(x)\tilde{E}(y) - \tilde{E_n}(y)\tilde{E}(x)|
$$

for another $C \geq 0$. Besides, for all $x, y \in \Omega$

$$
|\tilde{E_n}(x)\tilde{E}(y) - \tilde{E_n}(y)\tilde{E}(x)| \leq \tilde{E_n}(x)|E_n(y) - E(y)| + \tilde{E_n}(y)|E_n(x) - E(x)|
$$

Thus, for all $x, y \in \Omega$

$$
|\mu_n(x) - \mu(x)| \leq C \left[ \tilde{E_n}(x)|E_n(y) - E(y)| + \tilde{E_n}(y)|E_n(x) - E(x)| \right]
$$

By integrating the previous inequality on $\Omega^2$, and using Hölder inequality, we get

$$
\|\mu_n - \mu\|_{L^1} \leq 2C|\Omega| \, \|\tilde{E_n}\|_{L^2} \, \|E_n - E\|_{L^2}
$$

which gives the $L^1$ convergence of $(\mu_n)$ to $\mu$.

□

Lemma 6.3. *Let $(\mu_n)$ be a sequence in $\mathcal{P}_c(\overline{\Omega})$ and $(w_n)$ be the sequence of associated minimizers in (23). If $\mu_n \to \mu$ for the weak-$\star$ topology then $(w_n)$ converges to $w_0$, the minimizer associated with $\mu$, in $(K_1, \|\cdot\|_{L^\infty})$.*

Proof. For any $w \in K_1$ and $\mu_1, \mu_2 \in \mathcal{P}(\Omega)$, we have

$$
|\Lambda_{\mu_1}(w) - \Lambda_{\mu_2}(w)| = \left| \int_\Omega R_\sigma(x,w)(d\mu_1(x) - d\mu_2(x)) \right| \leq (\|w\|_{L^\infty} + \sigma \ln(2) + \|\nabla c\|_{L^\infty}) d_1(\mu_1, \mu_2)
$$

which comes from the fact that the map $\Omega \ni x \mapsto R_\sigma(x,w)$ is uniformly bounded by $\|w\|_\infty + \sigma \ln(2)$ and is $\|\nabla c\|_{L^\infty}$-Lipschitz. Then by compactness of $K_1$,

$$
\min_{K_1} \Lambda_{\mu_n} \to \min_{K_1} \Lambda_\mu
$$

and there exists $\tilde{w} \in K_1$ such that, up to the extraction of a subsequence, $w_n \to \tilde{w}$ in $(K_1, \|\cdot\|_{L^\infty})$. Therefore

$$
\begin{aligned}
\left| \min_{K_1} \Lambda_{\mu_n} - \Lambda_\mu(\tilde{w}) \right| &= \left| \Lambda_{\mu_n}(w_n) - \Lambda_\mu(\tilde{w}) \right| \\
&\leq |\Lambda_{\mu_n}(w_n) - \Lambda_\mu(w_n)| + |\Lambda_\mu(w_n) - \Lambda_\mu(\tilde{w})| \\
&\leq (\|w_n\|_{L^\infty} + \sigma \ln(2) + \|\nabla c\|_{L^\infty}) d_1(\mu_n, \mu) + |\Lambda_\mu(w_n) - \Lambda_\mu(\tilde{w})|
\end{aligned}
$$

which goes to zero by continuity of $\Lambda_\mu$. This ensures that $\min_{K_1} \Lambda_{\mu_n}$ converges to $\Lambda_\mu(\tilde{w})$ when $n$ goes to $+\infty$. The uniqueness of the limit ensures that

$$\Lambda_\mu(\tilde{w}) = \min_{K_1} \Lambda_\mu$$

From Lemma 3.2, there is a unique solution to (23), namely $w_0$. Hence $\tilde{w} = w_0$. □

We are now able to prove continuity of $\mathcal{Y}_1$ on $(K_1, \|\cdot\|_{L^\infty})$.

If $(w_n)$ converges to $w$ in $(K_1, \|\cdot\|_{L^\infty})$, by Lemma 6.2 $(\mu(w_n, E))$ converges to $\mu(w, E)$ in $L^1(\Omega)$, and therefore weakly converges to the same limit. Then, by Lemma 6.3, $(\mathcal{Y}_1(w_n, E))$ uniformly converges to $\mathcal{Y}_1(w, E)$.

Similarly, if $(E_n)$ converges to $E$ in $(K_2, \|\cdot\|_{L^2})$, by Lemma 6.2 $(\mu(w, E_n))$ converges to $\mu(w, E)$ in $L^1(\Omega)$, and therefore weakly converges to the same limit. Then, by Lemma 6.3, $(\mathcal{Y}_1(w, E_n))$ uniformly converges to $\mathcal{Y}_1(w, E)$.

*Continuity of the map $\mathcal{Y}_2$ on $(K_2, \|\cdot\|_{L^2})$.* We first need to prove the following preliminary results, which show continuity of the source term $f_{w,E}$ with respect to $w$ and $E$, and continuity of the solutions to the scalar transport equation with respect to the source term.

LEMMA 6.4.

*(1) Let $E \in K_2$, and $(w_n)$ be a sequence in $K_1$. If $\|w_n - w\|_{L^\infty} \longrightarrow 0$ for some $w \in K_1$, then $\|f_{w_n,E} - f_{w,E}\|_{L^\infty} \longrightarrow 0$.*

*(2) Let $w \in K_1$, and $(E_n)$ be a sequence in $K_2$. If $\|E_n - E\|_{L^2} \longrightarrow 0$ for some $E \in K_2$, then $\|f_{w,E_n} - f_{w,E}\|_{L^\infty} \longrightarrow 0$.*

PROOF.

(1) Let us denote $f_n := f_{w_n,E}$, $f := f_{w,E}$, $\mu_n := \mu(w_n, E)$ and $\mu := \mu(w, E)$. Let $z \in \overline{\Omega}$. We have

$$\begin{aligned} |f_n(z) - f(z)| &\leq \int_{\Omega^2} \delta^{-1} \left|G_\sigma(x, y, w_n)\mu_n(x) - G_\sigma(x, y, w)\mu(x)\right| dxdy \\ &\leq \int_{\Omega^2} \delta^{-1} \left|\mu_n(x) - \mu(x)\right| G_\sigma(x, y, w_n)\, dxdy \\ &+ \int_{\Omega^2} \delta^{-1} \left|G_\sigma(x, y, w_n) - G_\sigma(x, y, w)\right| \mu(x)\, dxdy \end{aligned}$$

Given that

- $\|\mu_n - \mu\|_{L^1} \longrightarrow 0$ (by Lemma 6.2)
- $|G_\sigma(x, y, w_n) - G_\sigma(x, y, w)| \underset{n\infty}{\longrightarrow} 0$ for all $x, y \in \Omega$,
- the function $G$ is bounded (for example, by $|\Omega|^{-1} e^{\frac{M_1 + \|c\|_\infty}{\sigma}}$ with $M_1$ given by (47)),

the right term then goes to zero as $n$ goes to infinity and is independent of $z$. Therefore,

$$\|f_{w_n,E} - f_{w,E}\|_{L^\infty} \longrightarrow 0$$

.

(2) Let us denote $f_n := f_{w,E_n}$, $f := f_{w,E}$, $\mu_n := \mu(w, E_n)$ and $\mu := \mu(w, E)$. Let $z \in \overline{\Omega}$. We have

$$|f_n(z) - f(z)| \leq \int_{\Omega^2} \delta^{-1} G_\sigma(x, y, w) \left|\mu_n(x) - \mu(x)\right| dxdy$$

The function $G$ is bounded and $\|\mu_n - \mu\|_{L^1} \longrightarrow 0$ by Lemma 6.2. Therefore, $\|f_{w,E_n} - f_{w,E}\|_{L^\infty} \longrightarrow 0$. □

LEMMA 6.5. *Let $f \in L^\infty(\Omega)$, and $u_f \in H_0^1(\Omega)$ be the unique solution to the following equation*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f(z), \ \forall z \in \Omega \\ u(s) = 0, \ \forall s \in \partial\Omega \end{cases}$$

*There exists a constant* $C(\Omega)$*, depending only on* $\Omega$*, such that*

$$\|u_f\|_{L^2} \leq C(\Omega)\|f\|_{L^\infty}$$

Proof. As $u_f$ is a weak solution, we have, for any $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_f \cdot \nabla v + \int_\Omega (\mathbf{V} \cdot \nabla u)\, v + \lambda \int_\Omega u_f\, v = \int_\Omega fv$$

Now, with $v = u_f$ we have, because $\lambda$ is positive, and using Hölder inequality

$$\|\nabla u_f\|_{L^2}^2 \leq \|f\|_{L^2}\, \|u_f\|_{L^2}$$

By Poincaré inequality

$$\|u_f\|_{L^2} \leq \frac{\pi}{\operatorname{diam}(\Omega)} \|\nabla u_f\|_{L^2}$$

Then

$$\|u_f\|_{L^2} \leq \frac{\pi}{\operatorname{diam}(\Omega)} \|f\|_{L^2} \leq \frac{\pi |\Omega|^{1/2}}{\operatorname{diam}(\Omega)} \|f\|_{L^\infty}$$

□

Corollary 6.1. *Let* $(f_n)$ *be a sequence in* $L^\infty$*. If* $\|f_n - f\|_{L^\infty} \longrightarrow 0$ *for some* $f \in L^\infty$*, then* $\|u_{f_n} - u_f\|_{L^2} \longrightarrow 0$ *in* $(K_2, \|\cdot\|_{L^2})$*, where* $u_{f_n}$ *is the unique solution to*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_n(z),\ \forall z \in \Omega \\ u(s) = 0,\ \forall s \in \partial\Omega \end{cases}$$

Proof. For every $n \in \mathbb{N}$, $u_{f_n} - u_f$ is solution to

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = (f_n - f)(z),\ \forall z \in \Omega \\ u(s) = 0,\ \forall s \in \partial\Omega \end{cases}$$

By Lemma 6.5, we have $C(\Omega)$ such that

$$\|u_{f_n} - u_f\|_{L^2} \leq C(\Omega)\, \|f_n - f\|_{L^\infty}$$

which gives the desired convergence. □

We are now able to prove continuity of $\mathcal{Y}_2$ on $(K_2, \|\cdot\|_{L^2})$.

If $(w_n)$ converges to $w$ in $(K_1, \|\cdot\|_{L^\infty})$, by Lemma 6.4, $f_{w_n,E}$ uniformly converges to $f_{w,E}$. Then, by Corollary 6.1, $(\mathcal{Y}_2(w_n, E))$ goes to $\mathcal{Y}_2(w, E)$ in $L^2(\Omega)$.

Similarly, if $(E_n)$ converges to $E$ in $(K_2, \|\cdot\|_{L^\infty})$, by Lemma 6.4, $f_{w,E_n}$ uniformly converges to $f_{w,E}$. Then, by Corollary 6.1, $(\mathcal{Y}_2(w, E_n))$ goes to $\mathcal{Y}_2(w, E)$ in $L^2(\Omega)$.

### Proof of Proposition 5.1

As the labour market reduces to one working place, the wage equilibrium is simply given by

$$w^* = L^{-1}(1)$$

Here, the wage can therefore be considered as exogenous, depending on the productivity of the firms at the unique working place of the city. Now, by Proposition 3.1, we have

$$\mu(x) = \frac{R(x, w^*)^{\frac{\theta}{1-\theta}}\, \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_0^1 R(y, w^*)^{\frac{\theta}{1-\theta}}\, \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\, dy},\ \forall x \in [0;1] \tag{50}$$

By equation (32),

$$\tilde{E}(x) = E_0 + \lambda^{-1} \frac{\int_0^x R(s, w^*)^{\frac{\theta}{1-\theta}}\, \tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\, ds}{\int_0^1 R(s, w^*)^{\frac{\theta}{1-\theta}}\, \tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\, ds}$$

Differentiating w.r.t $x \in [0;1]$, we obtain the following differential equation

$$\tilde{E}'(x) = \lambda^{-1} \frac{R(x,w^*)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}} \tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\, ds}$$

which, together with the boundary conditions $\tilde{E}(0) = E_0$ and $\tilde{E}(1) = E_0 + \lambda^{-1}$, admits the following unique solution

$$\tilde{E}(x) = \left[ E_0^{\frac{1+\gamma-\theta}{1-\theta}} + \left( \left(E_0 + \lambda^{-1}\right)^{\frac{1+\gamma-\theta}{1-\theta}} - E_0^{\frac{1+\gamma-\theta}{1-\theta}} \right) \frac{\int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds} \right]^{\frac{1-\theta}{1+\gamma-\theta}}$$

Now, let $\beta := (1+\gamma-\theta)/(1-\theta)$, and $\varphi(x) := \int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds / \int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}}\, ds$. From (34), the derivative of $\tilde{E}(x)$ with respect to $\beta$ has the same sign as

$$F(x) = \frac{\ln(E_0)E_0^\beta + \left(\ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta\right)\varphi(x)}{E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)} - \ln\left(E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)\right)$$

We have $F(0) = (1-\beta)\ln(E_0)$, and for all $x \in [0;1]$, $F'(x)$ has the same sign as

$$\varphi'(x) \left[ \ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta - \frac{\ln(E_0)E_0^\beta + \left(\ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta\right)\varphi(x)}{E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)} - 1 \right]$$

which is equivalent, as $\lambda$ goes to zero, to

$$\varphi'(x) \left[ \ln(\lambda^{-1})\lambda^{-\beta} - \frac{\ln(E_0)E_0^\beta + \ln(\lambda^{-1})\lambda^{-\beta}\varphi(x)}{E_0^\beta + \lambda^{-\beta}\varphi(x)} \right]$$

$\varphi'(x)$ is positive, and the second multiplicative term is also positive if $\lambda$ is small enough. Therefore, for all $x \in [0;1]$, $F(x) > 0$ and then $d\tilde{E}(x)/d\beta > 0$. As $\beta$ increases with $\gamma$, this means that $d\tilde{E}(x)/d\gamma > 0$.

### Proof of Proposition 5.2

Applying the weak formulation of pollution dispersion, with a constant unit test function $v := 1$, we get

$$\int_\Omega (\mathbf{V} \cdot \nabla E) + \lambda \int_\Omega E = \int_\Omega f_{\mu,w}$$

But

$$\int_\Omega (\mathbf{V} \cdot \nabla E) = -\int_\Omega E(\nabla \cdot \mathbf{V}) + \int_{\partial\Omega} E(s)\mathbf{V}(s) \cdot \mathbf{n}\, ds = 0$$

by the divergence theorem and equations satisfied by $E$ and $\mathbf{V}$. Therefore

$$\int_\Omega E = \lambda^{-1} \int_\Omega f_{\mu,w} = \lambda^{-1} \int_{\overline{\Omega}^2} |x-y|\, \mu(x)\, G_\sigma(x,y,w)\, dx\, dy$$

or, given that $m(x,y) := \mu(x)G_\sigma(x,y,w)$ is a probability density on $\Omega^2$,

$$\int_\Omega E = \lambda^{-1}\mathbb{E}\left[|X-Y|\right]$$

where the couple $(X,Y)$ follows the joint distribution of density $m(x,y)$.

**Proof of Proposition 5.3**

By equation (14), we have

$$\lambda \int_\Omega E\mu = \int_\Omega (\Delta E + f_{\mu,w} - \mathbf{V}\cdot\nabla E)\mu$$

First, $\mu$ is differentiable (because $w$ is, from Lemma 6.1), and we have, thanks to the divergence theorem

$$\int_\Omega \Delta E\mu = -\int_\Omega \nabla E\cdot\nabla\mu$$

with

$$\nabla\mu = \frac{\theta}{1-\theta}\nabla R_\sigma(\cdot,w)\,\mu - \frac{\gamma}{1-\theta}\nabla E\,\mu$$

Thus

$$-\int_\Omega \nabla E\cdot\nabla\mu = -\frac{\theta}{1-\theta}\int_\Omega (\nabla E\cdot\nabla R_\sigma(\cdot,w)\,\mu) + \frac{\gamma}{1-\theta}\int_\Omega |\nabla E|^2\,\mu$$

Now, using the divergence theorem, boundary conditions on $E$, and continuity equation on $\mathbf{V}$, we obtain

$$-\int_\Omega \mathbf{V}\cdot\nabla E\,\mu = \int_\Omega E\,[\nabla\cdot(\mathbf{V}\mu)] = \int_\Omega E\,[\nabla\mu\cdot\mathbf{V} + (\nabla\cdot\mathbf{V})\mu] = \int_\Omega E\,\nabla\mu\cdot\mathbf{V}$$

Thus

$$-\int_\Omega \mathbf{V}\cdot\nabla E\,\mu = \frac{\theta}{1-\theta}\int_\Omega ([\mathbf{V}\cdot\nabla R_\sigma(\cdot,w)]\,E\,\mu) - \frac{\gamma}{1-\theta}\int_\Omega \mathbf{V}\cdot\nabla E\,\mu$$

which yields

$$-\int_\Omega \mathbf{V}\cdot\nabla E\,\mu = \frac{\theta}{1-\theta-\gamma}\int_\Omega [\mathbf{V}\cdot\nabla R_\sigma(\cdot,w)]\,E\,\mu$$

Finally, injecting (6) and (6) into (6), and rearranging terms, yields

$$\lambda\int_\Omega E\mu = \int_\Omega f_{\mu,w}\,\mu \;-\; \int_\Omega \nabla E\cdot\nabla\mu \;+\; \frac{\theta}{1-\theta+\gamma}\int_\Omega [\mathbf{V}\cdot\nabla R_\sigma(\cdot,w)]\,E\,\mu$$

**Proof of Proposition 5.4**

The proof follows [Achdou et al., 2022], subsection 3.1. Fix any $\mu \in \mathcal{P}_c(\overline{\Omega})$. By Proposition 3.2, if $w \in C(\overline{\Omega},\mathbb{R}_+^*)$ is solution to (12) then $w$ is the unique minimizer of

$$\min_{w\in C(\overline{\Omega})} \left\{ \int_{y\in\Omega} \nu(y)\,[f(\ell(w(y))) - \ell(w(y))w(y)]\;dy + \int_{\overline{\Omega}} R_\sigma(x,w)d\mu(x) \right\} \tag{51}$$

whose Fenchel-Rockafellar dual problem writes

$$\sup_{l(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} \nu(y) f\left(\nu(y)^{-1} l(y)\right)\;dy - C_\sigma(l) \right\} \tag{52}$$

where

$$C_\sigma(l) := \sup_{w(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} l(y)w(y)\,dy - \int_{\overline{\Omega}} R_\sigma(x,w)d\mu(x) \right\} \tag{53}$$

By Fenchel-Rockafellar theorem, the supremum in (52) is attained for a certain $l^* \in C(\overline{\Omega})$ and

$$\min_{w\in C(\overline{\Omega})} \left\{ \int_{y\in\Omega} \nu(y)\,[f(\ell(w(y))) - \ell(w(y))w(y)]\;dy + \int_{\overline{\Omega}} R_\sigma(x,w)d\mu(x) \right\} = \max_{l(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} \nu(y) f\left(\nu(y)^{-1} l(y)\right)\right.$$

A necessary condition for $C_\sigma(l^*)$ not being equal to infinity is $l^* \geq 0$ and $\int_{\overline{\Omega}} l^*(y)\, dy \leq 1$. In this case, optimality conditions for (53) yield

$$l^*(y) = \int_{\overline{\Omega}} G_\sigma(x, y, w)\, d\mu(x),\ \forall y \in \overline{\Omega}$$

Now, consider the map $\Theta : C(\overline{\Omega}) \to C(\overline{\Omega})$ that associates, for any $w \in C(\overline{\Omega})$, the function $l$ defined by $y \mapsto \int_{\overline{\Omega}} G_\sigma(x, y, w)\, d\mu(x)$. By Proposition 3.2, $\Theta$ is a bijection. Thus problem (52) is equivalent to

$$\max_{l(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} [\nu(y) f\left(\nu(y)^{-1} l(y)\right) - l(y)\Theta^{-1}(l)(y)]\, dy + \int_{\overline{\Omega}} R_\sigma(x, \Theta^{-1}(l)) d\mu(x) \right\}$$

which is equivalent to

$$\max_{w(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} [\nu(y) f\left(\nu(y)^{-1} \Theta(w)(y)\right) - \Theta(w)(y)w(y)]\, dy + \int_{\overline{\Omega}} R_\sigma(x, w) d\mu(x) \right\}$$

i.e.

$$\max_{w(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} [\nu(y) f\left(\nu(y)^{-1} l_{\mu,w}(y)\right) - l_{\mu,w}(y)w(y)] dy + \int_{\overline{\Omega}} R_\sigma(x, w) d\mu(x) \right\}$$

Thus

$$\begin{aligned}
&\min_{w\in C(\overline{\Omega})} \left\{ \int_{y\in\Omega} \nu(y)\, [f(\ell(w(y))) - \ell(w(y))w(y)]\ dy + \int_{\overline{\Omega}} R_\sigma(x, w) d\mu(x) \right\} \\
&= \max_{w(\cdot)\in C(\overline{\Omega})} \left\{ \int_{\overline{\Omega}} [\nu(y) f\left(\nu(y)^{-1} l_{\mu,w}(y)\right) - l_{\mu,w}(y)w(y)] dy + \int_{\overline{\Omega}} R_\sigma(x, w) d\mu(x) \right\} \qquad (54)
\end{aligned}$$

Let $w_0$ be the unique solution to (12): for all $y \in \overline{\Omega}$, $\nu(y)\ell(w_0) = l_{\mu,w_0}(y)$. Hence

$$\begin{aligned}
\int_{y\in\Omega} \nu(y)\, [f(\ell(w_0(y))) - \ell(w_0(y))w_0(y)]\ dy + \int_{\overline{\Omega}} R_\sigma(x, w_0) d\mu(x) = \int_{\overline{\Omega}} [\nu(y) f\left(\nu(y)^{-1} l_{\mu,w_0}(y)\right) - l_{\mu,w_0}(y)w_0( \\
+ \int_{\overline{\Omega}} R_\sigma(x, w_0) d\mu(x)
\end{aligned}$$

which, together with equality (54), ensures that $w_0$ achieves the supremum in (38).

### Proof of Proposition 5.5

Condition (15) is equivalent to the mean-field equation

$$\int_{\overline{\Omega}} (x, \mu_{ac}(x)) d\mu(x) = \sup_{m\in(\overline{\Omega})} \int_{\overline{\Omega}} (x, m_{ac}(x)) dm(x),$$

where for every $x \in \overline{\Omega}$,

$$(x, \mu) = \begin{cases} \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}}, & \text{if } \mu \in_{ac} (\overline{\Omega}), \\ -\infty, & \text{otherwise.} \end{cases}$$

One can rewrite this equilibrium condition as follows:

$$\forall m \in (\overline{\Omega}), \quad \int_{\overline{\Omega}} (x, \mu) d(m - \mu)(x) \leq 0.$$

We recognize the first order condition of the following maximisation problem:

$$\sup_{\mu\in(\overline{\Omega})} \int_{\overline{\Omega}} (x, \mu) dx, \qquad (55)$$

where  is the potential of the game. In this setting,  represents the derivative (in the sense of measures) of , defined by

$$(\mu) = \begin{cases} \theta^{\theta-1}\left(R_\sigma(x,w)\mu(x)\right)^{\theta}\tilde{E}(x)^{-\gamma}, & \text{if } \mu \in_{ac} (\overline{\Omega}), \\ -\infty, & \text{otherwise.} \end{cases}$$

Since the utility function is a power function of the density of workers' residences, we observe that (55) is equivalent to

$$\sup_{\mu\in(\overline{\Omega})} \int_{\overline{\Omega}} (x,\mu)\, d\mu(x).$$

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